KIRISH

Masalaning qo‘yilishi va uning dolzarbligi. Yurtimiz istiqlolga erishgan ilk kunlardanoq, davlatimiz tomonidan amalga oshirilayotgan bunyodkorlik ishlari vatanimiz mustaqilligi va ozodligi tufaylidir. Mustaqillik zamirida yuz berayotgan islohatlar sezilarli darajada insoniyat turmush tarzini o’zgartirib yubordi. So‘ngi yillarda yoshlarga yaratilgan imkoniyatlar har bitta yigit qizni harakatdan to‘xtamaslikka undaydi.

Hozirgi kunda Prezidentimiz Sh.Mirziyoyev tomonidan “Mening nazarimda, jamiyat hayotining tanasi iqtisodiyot bo’lsa, uning joni va ruhi – ma’naviyatdir. Biz yangi O‘zbekistonni barpo etishda ana shu ikkita mustahkam ustunga, ya’ni, bozor tamoyillariga asoslangan kuchli iqtisodiyotga hamda ajdodlarimizning boy merosi, milliy va umuminsoniy qadriyatlarga asoslangan kuchli ma’naviyatga tayanamiz”[1-4]. Yoshlarga keng imkoniyatlar yaratib berilmoqda, katta – katta loyihalar ustida ishlanmoqda. Ularning bilim va istedodlarini shakllantirib, milliy ma’naviyatimizni uzoqlashib ketayotgani sezilib qolmoqda. Ular o‘zlari o‘qib kelgan xorijiy davlatlardagi tajribani o‘rganib, tajriba almashib kelishmoqda. Yangi O‘zbekiston taraqqiyot strategiyasining maqsadi – aholining barcha qatlamlariga munosib hayot darajasini va turmush sharoitlarini yaratib berish, ishtimoiy himoya va bandlikni ta’minlash, daromadlar barqaror o’sishiga erishish, jamiyatning madaniy darajasi, bag’rikenglik va mehribonlik fazilatlarini yanada mustahkamlashdan iborat [1-3].

O‘zbekistonda ta’lim tizimini isloh qilishning dasturiy hujjatlarida takidlanganidek, mamlakatimiz ta’lim tizimi hodimlari oldiga raqobatbardosh kadrlar tayyorlash, ta’lim tarbiya jarayonini jahon andozalari darajasiga yetkazishni asosiy vazifa qilib qo‘ygan[1]. Shu ma’noda olib qaraganda, yoshlarning yangi avlodi istiqbol masalalarini kun tartibiga dadil qo’yadigan va uni yecha oladigan, siyosiy hamda ijtimoiy – iqtisodiy hayotda o‘ziga mustaqil yo‘l topa oladigan qobiliyatga ega bo‘lishi kerak.

O‘zbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi “2020-2022-yillarda iqtisodiyot tarmoqlari va ijtimoiy soha uchun matematika bo’yicha oliy malakali kadrlar tayyorlash chora – tadbirlari dasturini ishlab chiqish to‘g‘risida” gi

Qarori, 2021-yil 19-yanvardagi 23-sonli “O‘zbekistonda yoshlarga oid davlat siyosatini 2025-yilgacha rivojlantirish konsepsiyasini tasdiqlash to‘g‘risida”gi Qarori bu boradagi amaliy ishlarning jadal suratlar bilan amalga oshirilayotganinini ko‘rsatadi. Oliy ma‘lumotli kadrlar sonini oshirish maqsadida 2022 yilda oliygohlar soni yana 5 taga oshirilib, 115 taga yetkazildi. 2021 yilda esa ularning soni 110 tani tashkil etgan. Bu haqida 2022 yil uchun ishlab chiqilgan budjetnomada ma’lumot berilgan [1-6]. So’ngi yillarda mamlakatimizda oliy ta’lim sifatini oshirishga qaratilgan bir qancha islohatlar amalga oshirilmoqda. Ushbu magistrlik dissertatsiyasi mavzusi ana shu talab va vazifalardan kelib chiqib tanlandi.

Tub sonlarning taqsimotini nazariy jihatdan tekshirgan matematiklardan biri rus matematigi P.L.Chebishevdir. U bu masalada katta muvaffaqiyatga erishdi. Tub sonlar sonlarning taqsimoti haqidagi ilk ma’lumotlarni birinchi marta 1849-yilda yozgan. P.L.Chebishev 1852-yilda bu masalani to‘liq yechdi, bundan tashqari P.L.Chebishev asarlarida $\pi(x)$ va boshqa sonli funksiyalarning xossalarini tekshirish uchun kuchli elementar metodlarni ko‘rsatib berdi. U x ning yetarlicha katta qiymatlarida $\pi(x)$ ni baholash uchun quyidagi tengsizlikni o‘rinli ekanini ko‘rsatib berdi

$$0,92129 < \frac{\pi(x)}{\frac{x}{\ln x}} < 1,10555.$$

b). Ishning maqsad va vazifalari. Magistrlik dissertatsiyasining asosiy maqsadi Chebishev funksiyalarini o‘rganib, Chebishev funksiyalariga ilgari olingan baholarni aniqlashtirishdan iborat.

Bu maqsadga erishish uchun quyidagi vazifalarni amalga oshirish kerak:

1) Chebishev funksiyalarini o‘rganish ;

2) Chebishev funksiyalariga ilgari olingan baholardan foydalanib yangi baho olish.

c). Tadqiqotning ob’yekti va predmeti bo‘lib ishda algebraik metodlar, Dirixle xarakterlari va sonlar nazariyasining ba’zi additiv masalalari, ularni yechish metodlari, olingan baholar tahlili hisoblanadi. Tadqiqotning predmeti bo‘lib, Chebishev funksiyalariga ilgari olingan baholardan foydalanib yangi baho olish hisoblanadi.

d). Tadqiqot natijalarining ilmiy jihatdan yangilik darajasi. Ishda $\psi(x,\chi)$ funksiya uchun baholar olingan. Bunday formulalar umuman olganda mavjud, lekin ularning qoldiq hadida "O"-simvoli ishtiroq etgani uchun ba'zi bir sonli hisoblashlar qatnashgan masalalarda foydalanib bo'lmaydi. Ushbu ishda biz $\psi(x,\chi)$ funksiya uchun "O"-simvoli ishtirok etmagan natija isbotlangan.

f). Ishning asosiy natijalari. Ushbu ishda biz $\psi(x,\chi)$ funksiya uchun "O"-simvoli ishtirok etmagan quydagi natija isbotlangan

χ $q$moduli bo'yicha Dirixle xarakteri va 3$\leq T \leq x$ bo'lsa.

U holda

$$\psi(x,\chi)=\delta_{\chi}x - E_{\tilde{\beta}}\frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} + R(x,T),$$

bu yerda $\delta_{\chi}\chi \neq \chi_0$ yoki$\chi = \chi_0$ (χ_0 $-$ bosh xarakter) bo'lishiga qarab 0 yoki 1

$$\delta_{\chi} = \begin{cases} 1, & \text{agar } \chi = \chi_0 - \text{bosh xarakter bo'lsa,} \\ 0, & \text{agar } \chi \neq \chi_0 - \text{bo'lsa} \end{cases},$$

$$E_{\tilde{\beta}} = \begin{cases} 1, & \text{agar } \chi = \tilde{\chi} - \text{maxsus haqiqiy xarakter bo'lsa,} \\ 0, & \text{agar } \chi \neq \tilde{\chi} - \text{ bo'lsa} \end{cases},$$

$\tilde{\beta}$ $-$maxsus$\chi = \tilde{\chi}$ $-$haqiqiy xarakterga mos haqiqiy nol bo'lib o'ng tomondagi yig'indi 0<$\sigma < 1, |\gamma| < T$ sohadagi maxsus noldan tashqari barcha $\rho = \beta + i\gamma$nollar bo'yicha olinadi. Formuladagi qoldiq had uchun quyidagi baho o'rinli:

$$|R(x,T)| < 1445{,}91\frac{x}{T}\log^2 qx + E_{\tilde{\beta}}x^{\frac{1}{4}}\log x +$$

$$+2{,}67(\log x)\min\left(1, \frac{x}{\pi < x > T}\right)$$

bu yerda $< x >$ bilan x dan unga eng yaqin tub sonning darajasigacha bo'lgan masofa belgilangan.

g). Tadqiqot natijalarining amaliy ahamiyati va tadbiqi. Dissertatsiya ishi ilmiy – nazariy xarakterda bo'lib, matematika, fizika va texnikaning ko'plab masalalari, juda ko'p amaliy va iqtisodiy masalalarni yechishda foydalanish mumkin. Shuningdek, shu sohada ilmiy izlanishlar olib boruvchi mutaxassislar va talabalarga maxsus kurs va seminarlar o'tishda foydalanish mumkin.

h). Dissertatsiya ishining ilmiy –tadqiqot ishlari rejalari bilan bog'liqligi. Dissertatsiyaning mavzusi Termiz davlat universiteti ilmiy kengashi tomonidan tasdiqlangan va Termiz davlat universiteti Algebra va geometriya kafedrasida olib borilayotgan ilmiy tadqiqot ishlari bilan bevosita bog'liq.

i). Natijalarning qo‘llanilishi. Dissertatsiya natijalaridan O‘zRFA ning Matematika instituti, Qarshi davlat universiteti, Andijon Mashinasozlik instituti va Termiz davlat universitetlarida ilmiy izlanishlar olib borayotgan mutaxassislar foydalanishi hamda talabalarga maxsus kurs va seminarlar o‘tishda foydalanish mumkin.

j). Ishning sinovdan o‘tishi. Ishning asosiy natijalari “Amaliy matematika va axborot texnologiyalarining zamonaviy muammolari” xalqaro ilmiy amaliy anjumani. 2022-yil 11-12 may. Buxoro (O‘zbekiston). Shuningdek Termiz davlat universiteti Algebra va geometriya kafedrasi hamda O‘zbekiston Respublikasi Fanlar akademiyasi matematika instituti hamkorligida bo‘lib o‘tgan “Algebra va analizning dolzarb masalalari” ilmiy-amaliy anjumani. 18-19 noyabr. 2022-yil Termiz (O‘zbekiston). Termiz davlat universiteti “Algebra va geometriya’’ kafedrasi qoshidagi fizika-matematika fanlari doktori, professor I.Allakov rahbarligidagi “Algebraning zamonaviy masalalari” ilmiy seminarlarida ma’ruza qilinib muhokama qilingan (2022-2023 yillar).

k). Natijalarning e’lon qilinganligi. Ish yuzasidan 2ta maqola chop etilgan [23-24].

I-BOB. CHEKLI TARTIBLI BUTUN FUNKSIYALAR

I.1-§. Cheksiz ko‘paytmalar. Yaqinlashuvchi va uzoqlashuvchi cheksiz ko‘paytmalar

Cheksiz ko‘paytmalar. Avvalo, kompleks o‘zgaruvchining funktsiyalari nazariyasiga doir ba’zi tushunchalarni keltiramiz. Funktsiyalarni tasvirlashda cheksiz qatorlar bilan birga cheksiz ko‘paytmalar ham muhim ashamiyatga ega. Kompleks o‘zgaruvchili golomorf funktsiyalarni bunday ko‘paytmalar yordamida ifodalashdan oldin cheksiz ko‘paytmalar va ularning yaqinlashish hamda uzoqlashishi aniqlashga qisqacha to‘xtalib o‘tamiz.

Ma’lumki, agar $f(z)$funksiyani a nuqtaning biror atrofida $z-a$ ga nisbatan darajali qatorga yoyish mumkin bo‘lsa, $f(z)$ funksiyani a nuqtada golomorf deyiladi.

G sohaning har bir nuqtasida golomorf bo‘lgan funksiyaga *G sohada golomorf funksiya* deyiladi.

1.1-ta’rif. $u_1, u_2, \dots, u_k, \dots$ lar (-1) ga teng bo‘lmagan kompleks sonlarning cheksiz ketma-ketligi bo‘lsa,

$$\prod_{k=1}^{\infty}(1+u_k) = (1+u_1)(1+u_2)\cdots(1+u_n)\cdots \qquad (1.1)$$

ko‘rinishdagi ifodaga *cheksiz ko‘paytma* deyiladi.

$$p_n = \prod_{k=1}^{n}(1+u_k) = (1+u_1)(1+u_2)\cdots(1+u_n) \qquad (1.2)$$

ko‘rinishdagi ifodaga *xususiy ko‘paytma* deyiladi.

n ga 1,2,3, . . . , qiymatlar berib noldan farqli kompleks sonlarning $p_1, p_2, \dots, p_n, \dots$ ketma-ketligini hosil qilamiz.

Yaqinlashuvchi va uzoqlashuvchi cheksiz ko‘paytmalar.

Bunda uchta hol bo‘lishi mumkin:

1). $p_1, p_2, \dots, p_n, \dots$ sonlar ketma-ketligi chekli, noldan farqli p soniga yaqinlashadi, ya’ni $\lim_{n\to\infty} p_n = p, (p \neq 0)$;

2). $p_1, p_2, \dots, p_n, \dots$ sonlar ketma-ketligi 0 soniga yaqinlashadi, ya’ni $\lim_{n\to\infty} p_n = 0$;

3) $p_1, p_2, \dots, p_n, \dots$sonlar ketma-ketligi uzoqlashuvchi, ya’ni hech qanday chekli limitga ega emas.

Birinchi holda (1.1)-cheksiz ko‘paytma yaqinlashuvchi deyiladi va p soniga bu ko‘paytmaning qiymati deyiladi.

Qolgan ikkala holda (1)- cheksiz ko‘paytma uzoqlashuvchi deyiladi.

1.1-misol

$$1 \cdot \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \, \cdots \, \cdot \frac{n^2}{n^2-1} \cdots$$

cheksiz ko‘paytmani qaraymiz.

Bunda

$$p_n = 1 \cdot \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \, \cdots \, \cdot \frac{n^2}{n^2-1} =$$

$$= 1 \cdot \frac{2^2}{(2-1)(2+1)} \cdot \frac{3^2}{(3-1)(3+1)} \cdot \, \cdots \, \cdot \frac{n^2}{(n-1)(n+1)} = \frac{2n}{n+1}$$

bo‘lgani uchun

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = 2.$$

demak, qaralayotgan cheksiz ko‘paytma yaqinlashuvchi[7].

1.2-misol. $1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \cdots \cdot \frac{1}{n} \cdots$ cheksiz ko‘paytmani qaraymiz.

Bunda

$$p_n = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \cdots \cdot \frac{1}{n} = \frac{1}{n!}$$

va

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{1}{n!} = 0.$$

Demak, bu yuqorida qaralgan 2-holga asosan uzoqlashuvchi.

1.3-misol. $2 \cdot \frac{1}{2} \cdot 3 \cdot \frac{1}{3} \cdot \cdots \cdot n \cdot \frac{1}{n} \cdots$cheksiz ko‘paytmani qaraymiz.

Bunda

$$p_n = 2 \cdot \frac{1}{2} \cdot 3 \cdot \frac{1}{3} \cdot \cdots \cdot n \cdot \frac{1}{n}$$

bo‘lib $n = 2k$ juft son bo‘lsa,

$$p_{2k} = 2 \cdot \frac{1}{2} \cdot 3 \cdot \frac{1}{3} \cdot \cdots \cdot k \cdot \frac{1}{k} = 1;$$

bo‘lib $n = 2k + 1$ toq son bo‘lsa,

$$p_{2k+1} = 2\cdot\frac{1}{2}\cdot 3\cdot\frac{1}{3}\cdot\cdots\cdot k\cdot\frac{1}{k}\cdot(k+1) = k+1$$

Demak, $\lim_{n\to\infty} p_n$ mavjud emas va bu yuqorida qaralgan 3-holga asosan uzoqlashuvchi.

Cheksiz ko‘paytmaning yaqinlashuvchi bo‘lishlik tushunchasini tengsizlik yordamida ham ifodalash.

Cheksiz ko‘paytmaning yaqinlashuvchi bo‘lishlik tushunchasini tengsizlik yordamida ham ifodalash mumkin.

Haqiqatan ham agar (1.1) cheksiz ko‘paytma biror p soniga yaqinlashsa,

$$p = (1+u_1)(1+u_2)\cdots(1+u_n)\cdots \qquad (1.3)$$

deb yoza olamiz.Bu holda n ning o‘sishi bilan $\frac{p}{p_n}$ nisbat birga intiladi:

$$\lim_{n\to\infty}\frac{p}{p_n} = p\cdot\lim_{n\to\infty}\frac{1}{p_n} = p\cdot\frac{1}{\lim\limits_{n\to\infty} p_n} = \frac{p}{p} = 1,\ \ (p\neq 0).$$

Aksincha, agar n ning o‘sishi bilan $\frac{p}{p_n}$ nisbat birga intilsa, u holda (1.1) ko‘paytma p soniga yaqinlashadi. Boshqacha so‘zlar bilan aytganda, agar ixtiyoriy yetarlicha kichik $\varepsilon > 0$ soni uchun shunday bir $N = N(\varepsilon)$ sonini topish mumkin bo‘lsaki, $n \geq N$ bo‘lganda

$$\left|\frac{p}{p_n} - 1\right| < \varepsilon$$

bajarilsa, (1.1) cheksiz ko‘paytmani $p(p\neq 0)$ soniga yaqinlashuvchi deyiladi.

Cheksiz ko‘paytmalar yaqinlashishining asosiy belgisi (zaruriy va yetarli shartlari).

Cheksiz ko‘paytmalar nazariyasida uning hadlarini (ko‘paytuvchilarini) bilgan holda ko‘paytmaning yaqinlashuvchi yoki uzoqlashuvchi ekanligini aniqlash asosiy masala hisoblanadi. Biz faqat kelgisi tekshirishlarimizda zarur bo‘lgan birta yaqinlashish belgisini qarash bilan chegaralanamiz. Bu belgi cheksiz ko‘paytmaning yaqinlashuvchi ekanligini unga mos qatorning yaqinlashuvchi ekanligidan aniqlash imkonini beradi.

1.1-teorema. Agar

$$u_1 + u_2 + \cdots + u_n + \cdots \qquad (1.4)$$

qator absolyut yaqinlashuvchi bo‘lsa, u holda

$$(1+u_1)(1+u_2)\cdots(1+u_n)\cdots \qquad (1.5)$$

ko‘paytma ham yaqinlashuvchi bo‘ladi.

Isboti. Teoremaning shartiga ko‘ra (1.4) qator absolyut yaqinlashuvchi, ya’ni $|u_1|+|u_2|+\cdots+|u_n|+\cdots$qator yaqinlashuvchi. Shuning uchun ham $\lim\limits_{n\to\infty}|u_n| = 0$ va biz umumiylikni chegaralamagan holda $|u_n| \le \frac{1}{2}$, $(n = 1,2,3,\dots)$ deb olishimiz mumkin.

Avvalo $u_n = a, (n = 1,2,3,\dots)$ haqiqiy son bo‘lsin. U holda

$$|\ln(1+u_n)| = \left|u_n - \frac{u_n^2}{2} + \frac{u_n^3}{3} - \cdots\right| \le |u_n| + \frac{|u_n|}{2}(|u_n| + |u_n^2| + \cdots) =$$

$$= |u_n| + \frac{|u_n|}{2}\cdot\frac{|u_n|}{1-|u_n|} \le |u_n| + \frac{|u_n|}{\cancel{2}}\cdot\cancel{2}\cdot\ |u_n| = |u_n| +\ |u_n^2|$$

$$< 2\cdot\ |u_n|.$$

Bu yerdan

$$\ln(1+u_1)+\cdots+\ln(1+u_n) = \ln(1+u_1)\cdots(1+u_n)$$

ketma-ketlikning yaqinlashishi kelib chiqadi. Haqiqatan ham

$$|\ln(1+u_1)+\cdots+\ln(1+u_n)+\cdots| \le 2(|u_1|+|u_2|+\cdots+|u_n|+\cdots).$$

O‘ng tomondagi qator shartga ko‘ra yaqinlashuvchi va demak, chap tomoni ham yaqinlashuvchi bo‘ladi. Bundan esa (1.5) ning xususiy ko‘paytmalaridan tuzilgan ketma-ketlikning yaqinlashuvchi va (1.5) ning ham yaqinlashuvchi ekanligi kelib chiqadi.

Endi $u_n, (n = 1,2,3,\dots)$-ixtiyoriy kompleks son bo‘lsin. Bu holda ikkita haqiqiy sonlar ketma-ketliklari

$$|v_n| = |(1+u_1)\cdots(1+u_n)| = |1+u_1|\cdots|1+u_n| \qquad (1.6)$$

$$arg v_n = arg(1+u_1)\cdots(1+u_n) = arg(1+u_1)+\cdots+arg(1+u_n) \qquad (1.7)$$

ning yaqinlashuvchi ekanliklarini ko‘rsatishimiz kerak.

Tushunarliki (1.6) ning yaqinlashuvchi bo‘lishi uchun $|v_n|^2, n = 1,2,3,\dots$ ketma-ketlikning yaqinlashuvchi bo‘lishi zarur va yetarlidir. Bu yerda

$$|1+u_n|^2 = |1+\alpha_n+i\beta_n|^2 = 1+\alpha_n^2+\beta_n^2+2\alpha_n;$$

$$u_n = \alpha_n + i\beta_n, \alpha_n, \beta_n \in R \text{va } |\alpha_n^2+\beta_n^2+2\alpha_n| \le |u_n|^2 + 2|u_n|$$

bo‘lgani uchun $|v_n|^2$ ning yaqinlashishi yuqorida isbotlanganidan kelib chiqadi[8].

(1.7) ning yaqinlashishi esa yetarlicha katta $n_0$va $n > n_0$lar uchun

$$|arg(1+u_n)| = \left|\arcsin\frac{\beta_n}{\sqrt{(1+\alpha_n)^2+\beta_n^2}}\right| < \frac{\pi}{2}\frac{\beta_n}{\sqrt{(1+\alpha_n)^2+\beta_n^2}} < \pi|\beta_n|$$

tengsizlikning o‘rinli ekanligidan keldib chiqadi.

$$(1+u_1)(1+u_2)\cdots(1+u_n)\cdots$$

ko‘paytmada barcha u_n larni bir xil ishorali haqiqiy sonlar deb hisoblab

$\sum_{n=1}^{\infty} u_n$ qatorning yaqinlashuvchi bo‘lishligi yuqoridagi ko‘paytmaning yaqinlashuvchi bo‘lishligining yetarli sharti bo‘libgina qolmasdan balki uning yaqinlashishining zaruriy sharti ham bo‘lishini osonlik bilan ko‘rsatish mumkin.

Haqiqatan ham, agar u_n sonlari musbat va qaralayotgan ko‘paytma yaqinlashuvchi bo‘lsa, u holda $p_n = (1+u_1)(1+u_2)\cdots(1+u_n)$sonlar ketma-ketligi o‘sib borib biror musbat o‘zgarmas son M dan kichik bo‘lib qolaverishi kerak. $p_n$ning ifodasidagi qavslarni ochib biz $u_1+u_2+\cdots+u_n$yig‘indining ham $n$ning qanday bo‘lishidan qat’iy nazar M dan kichik bo‘lishi kerak ekanligini ko‘ramiz. Bu esa $\sum_{n=1}^{\infty} u_n$ qator yaqinlashuichi deganidir.

Agar $u_n = -c_n$ sonlari manfiy bo‘lsalar, u holda $\sum_{n=1}^{\infty} s_n$ qatorni uzoqlashuvchi deb olib $\prod_{n=1}^{\infty}(1-c_n)$ ko‘paytmaning ham uzoqlashuvchi ekanligini ko‘rsatamiz.

Haqiqatan ham, n ning cheksiz o‘sishi bilan $lnp_n = ln(1-c_1)+\cdots+ ln(1-c_n)$ ifoda $-\infty$ ga intiladi. Chunki umumiy hadi $ln(1-c_n) = -\frac{c_n}{1-vc_n}$ bo‘lgan qator uzoqlashuvchi (biz bu yerda n ning biror qiymatidan boshlab$c_n < 1$ deb hisoblaymiz, aks holda p_n sonlari orasida cheksiz ko‘p musbat va manfiylari bo‘lgani uchun ko‘paytmanig uzoqlashuvchi ekanligi o‘z-o‘zidan tushunarli bo‘ladi). Bu yerdan n ning cheksiz o‘sishi bilan $p_n$ning nolga intilishi kelib chiqadi va demak, bizning ko‘paytmamiz uzoqlashuvchi. Agar

$$(1+|u_1|)(1+|u_2|)\cdots(1+|u_n|)\cdots \quad (1.5')$$

ko‘paytma yaqinlashuvchi bo‘lsa, (1.5) ko‘paytmani absolyut yaqinlashuvchi deb atashga shartlashib olamiz. Isbotlanganiga asosan (1.5′) cheksiz ko‘paytmaning yaqinlashishi

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots \quad (1.8)$$

qatorning yaqinlashishiga ekvivalent. Shunday qilib absolyut yaqinlashuvchi ko‘paytma (1.5)ni bizning aniqlashimizda (1.5′) ko‘paytmaning yaqinlashuvchi bo‘lishlik talabini (1.8)- qatorning yaqinlashuvchi bo‘lishlik talabi bilan almashtirish mumkin.

Shartli yaqinlashuvchi ko‘paytmalar. Golomorf funksiyalarni cheksiz ko‘paytmalar yordamida ifodalash.

Yuqorida isbotlangan teorema bizni har qanday absolyut yaqinlashuvchi ko‘paytma yaqinlashuvchi ko‘paytmadan iborat degan xulosaga olib keladi. Teskari tasdiqni o‘rinli deyish xato bo‘lar edi, ya’ni shunday yaqinlashuvchi ko‘paytmalar borki, ular absolyut yaqinlashuvchi ko‘paytma bo‘lmaydi. Bunday ko‘paytmalarni shartli yaqinlashuvchi ko‘paytmalar deyiladi.

Shartli yaqinlashuvchi ko‘paytmaga misol keltiramiz.

1.4-misol. Ushbu ko‘paytma

$$(1+1)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{6}\right)\cdots$$

ni qaraymiz. Bu yerda

$$p_n = \begin{cases} \frac{n+1}{n}, agar\ n\ toq\ son\ bo‘lsa; \\ 1, \qquad agar\ n\ juftson\ bo‘lsa \end{cases}$$

bo‘lgani uchun $\lim_{n\to\infty} p_n = 1$ va qaralayotgan ko‘paytma yaqinlashuvchi. Ikkinchi tomondan esa

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad (1.8')$$

qator uzoqlashuvchi bo‘lgani uchun ham qaralayotgan ko‘paytma absoyut yaqinlashuvchi emas[9].

Golomorf funksiyalarni cheksiz ko‘paytmalar yordamida ifodalash.

Bizga

$$[1 + u_1(z)][1 + u_2(z)]\cdots[1 + u_n(z)]\cdots \quad (1.9)$$

cheksiz ko‘paytma berilgan bo‘lsin. Bu ko‘paytmadagi barcha $u_n(z)$lar biror G

sohadagi golomorf funksiyalar va (1.9) dagi barcha ko‘paytuvchilar G sohaning ixtiyoriy z nuqtasida noldan farqli. z ning G sohadagi ixtiyoriy qiymatida

$$|u_n(z)| < a_n \tag{1.10}$$

tengsizlik o‘rinli deb qaraymiz va sonli qator

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{1.11}$$

yaqinlashuvchi bo‘lsin. Bu holda yuqorida isbotlanganiga asosan (1.9) ko‘paytma G sohaning ixtiyoriy z nuqtasida yaqinlashuvchi va u G sohaning birorta ham z nuqtasida nolga aylanmaydigan kompleks o‘zgaruvchining biror $f(z)$ funksiyasini ifodalaydi. $f(z)$ funksiyasining G sohadagi golomorf funksiya ekanligini isbotlaymiz.

Haqiqatan ham buning uchun

$$f_n(z) = [1 + u_1(z)][1 + u_2(z)]\cdots[1 + u_n(z)]$$

deb olsak, Veyershtrassning 1-teoremasiga ko‘ra G sohadagi golomorf funksiyalar $f_n(z)$ ketma-ketligining shu sohada $f(z)$funksiyaga tekis yaqinlashishini ko‘rsatishimiz yetarli bo‘ladi.

Har bir hadi biror G sohada analitik funksiya bo‘lgan

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots \tag{1.4’}$$

cheksiz qator berilgan bo‘lsin. $(1.1') -$qator G sohaning har bir z nuqtasida yaqinlashuvchi bo‘lsin. Uning yig‘indisini $f(z)$ orqali belgilaylik. Qanday shartda analitik funksiyalarning yaqinlashuvchi qatorining yig‘indisining o‘zi ham analitik funksiya bo‘ladi.

Bunday shart (1.4’) qatorning G sohada (yoki hech bo‘lmaganda G sohaga to‘liq tegishli bo‘lgan $\overline{G'}$ yopiq sohada) tekis yaqinlashuvchi bo‘lishlik shartidir. Bunga ikkita qismdan iborat bo‘lgan quyidagi Veyershtrass teoremasi javob beradi.

1.2-teorema (Veyershtrassning 1-teoremasi)**.** Agar (1.4’) qator hech bo‘lmaganda G sohaga to‘liq tegishli bo‘lgan $\overline{G'}$ yopiq sohada tekis yaqinlashuvchi bo‘lsa:

Birinchidan, $(1.4') -$qatorning yig‘indisi $f(z)$ G sohada analitik funksiyani ifodalaydi.

Ikkinchidan, (1.4′) qatorni istalgan marta differensiallash natijasida hosil bo‘lgan yangi qator ham $\overline{G'}$ yopiq sohada tekis yaqinlashuvchi bo‘lib $f(z)$ ni mos ravishda differensiallash natijasida hosil bo‘lgan funksiyani ifodalaydi. Qisqa qilib aytganda (1.4′) qatorni istalgan marta hadlab differensiallash mumkin.

Quyidagicha belgilash kiritamiz:

$$(1+a_1)(1+a_2)\cdots(1+a_n)=p_n\,,(1+a_1)(1+a_2)\cdots(1+a_n)\cdots=p$$

Bu ko‘paytmalar yuqorida isbotlanganiga asosan yaqinlashuvchi. Endi $f(z)-f_n(z)$ ayirmaning modulini baholaymiz.

$$|f(z)-f_n(z)|=|f_n(z)|\left|\frac{f(z)}{f_n(z)}-1\right|\le p_n\left|\frac{f(z)}{f_n(z)}-1\right|,$$

bunda biz

$$|f_n(z)|\le[1+|u_1(z)|][1+|u_2(z)|]\cdots[1+|u_n(z)|]<$$
$$<(1+a_1)(1+a_2)(1+a_n)=p_n$$

ekanligidan foydalandik. Ikkinchi tomondan esa G sohadagi z nuqta qanday bo‘lishidan qat’iy nazar ixtiyoriy n uchun

$$\left|\frac{f(z)}{f_n(z)}-1\right|\le\frac{p}{p_n}-1 \qquad (1.12)$$

bajariladi. Haqiqatan ham,

$$\frac{f_{n+k}(z)}{f_n(z)}-1=[1+u_{n+1}(z)][1+u_{n+2}(z)]\cdots[1+u_{n+k}(z)]-1=u_{n+1}(z)+$$
$$u_{n+2}(z)+u_{n+k}(z)+u_{n+1}(z)u_{n+2}(z)+\cdots+u_{n+1}(z)u_{n+2}(z)\ldots u_{n+k}(z)$$

bo‘lganidan

$$\left|\frac{f_{n+k}(z)}{f_n(z)}-1\right|<a_{n+1}+a_{n+2}+\cdots+a_{n+k}+a_{n+1}a_{n+2}+\cdots+a_{n+1}a_{n+2}\cdots a_{n+k}$$
$$=\frac{p_{n+k}}{p_n}-1.$$

Oxirgi tengsizlik ixtiyoriy k uchun o‘rinli. Bunda $k\to\infty$ da limitga o‘tib (1.12)-munosabatni hosil qilamiz.

(1.12) ni inobatga olgan holda (1.11) ni G sohadagi z nuqta qanday bo‘lishidan qat’iy nazar $n\ge N(\varepsilon)$ bo‘lganda quyidagicha yozishimiz mumkin:

$$|f(z)-f_n(z)|<p_n\left(\frac{p}{p_n}-1\right)=p-p_n<\varepsilon.$$

Bu oxirgi tengsizlik G sohadagi $f_n(z)$ golomorf funksiyalar ketma-ketligining $f(z)$ funksiyaga tekis yaqinlashishini isbotlaydi.

I.2 Chekli tartibli butun funksiyalar. Ularni cheksiz ko‘paytmalar yordamida ifodalash, Veyershtrass formulasi

1.2-ta’rif. Kompleks $s-$tekislikning har qanday chekli qismida analitik bo‘lgan $f(s)$ funksiyaga butun funksiya deyiladi.

Biz quyida ikkita teoremani keltiramiz.Ulardan biri nollar faqat berilgan cheksiz ketma-ketlikdagi sonlar bo‘lgan butun funksiyaning mavjudligi haqida, ikkinchisi esa butun funksiyani nollari bo‘yicha cheksiz ko‘paytmaga yoyish haqidadir.Bu keyingi teorema algebraning asosiy teoremasining umumlashmasidan iboratdir.

1.3-teorema. Agar

$$a_1,\ a_2,\dots,a_n,\dots \qquad (1.13)$$

lar kompleks sonlarning modullari o‘sib borish

$$0<|a_1|\le|a_2|\le\dots\le|a_n|\le\dots$$

tartibida joylashtirilgan cheksiz ketma-ketligi bo‘lib va

$$\lim_{n\to\infty}\frac{1}{|a_n|}=0 \qquad (1.14)$$

bo‘lsa, u holda nollari faqat a_n lardan iborat bo‘lgan $G(s)$ butun funksiya mavjud (agar a_n lar orasida o‘zaro tenglari mavjud bo‘lsa, $G(s)$ ning nollari ham karrali bo‘ladi).

Isboti. Agar (1.13) dagi sonlardan ba’zilari bir xil modulga ega bo‘lsa, biz bu sonlarni ixtiyoriy tartibda joylashtiramiz. (1.14)- shartdan va limitning ta’rifidan tushunarliki, ixtiyoriy yetarlicha katta R uchun moduli R dan kichik bo‘lgan chekli sondagi a_n miqdorlar mavjud. Ana shuni ta’kidlagan holda biz nollarifaqatva faqat a_n lardan iborat bo‘lgan $G(s)$ butun funksiyani qurish mumkin ekanligini isbotlaymiz.

Teoremaning isbotlashda avvalo a_n sonlari orasida nolga tenglar yo‘q deb qaraymiz. Ushbu

$$u_v = \left(1 - \frac{z}{a_v}\right) e^{\frac{z}{a_v} + \frac{1}{2}\left(\frac{z}{a_v}\right)^2 + \frac{1}{3}\left(\frac{z}{a_v}\right)^3 + \cdots + \frac{1}{v-1}\left(\frac{z}{a_v}\right)^{v-1}} \quad (1.15)$$

ifodani qaraymiz. $|z| < |a_v|$deb faraz qilsak, (1.15) dan

$$lnu_v = ln\left(1 - \frac{z}{a_v}\right) + \frac{z}{a_v} + \frac{1}{2}\left(\frac{z}{a_v}\right)^2 + \frac{1}{3}\left(\frac{z}{a_v}\right)^3 + \cdots + \frac{1}{v-1}\left(\frac{z}{a_v}\right)^{v-1}$$

ga ega bo‘lamiz. Bunda biz $ln\left(1 - \frac{z}{a_v}\right)$ ni markazi nol nuqtada, radiusi $|a_v|$ ga teng bo‘lgan doiraning ichida bir qiymatli golomorf funksiya deb qarab va $z = 0$ da nolga teng deb hisoblaymiz. Shularga asoslanib

$$lnu_v = -\frac{1}{v}\left(\frac{z}{a_v}\right)^v - \frac{1}{v+1}\left(\frac{z}{a_v}\right)^{v+1} - \cdots \quad (1.16)$$

deb yoza olamiz. Bundan

$$u_v = e^{-\frac{1}{v}\left(\frac{z}{a_v}\right)^v - \frac{1}{v+1}\left(\frac{z}{a_v}\right)^{v+1} - \cdots} \quad (1.15')$$

Endi biz

$$u_1 \cdot u_2 \cdot \cdots \cdot u_v \cdots \quad (1.17)$$

cheksiz ko‘paytmaning kompleks tekislikning ixtiyoriy z, $(z \neq a_v)$nuqtasida yaqinlashuvchi vanollari $a_1, a_2, \dots, a_n, \dots$ lar bo‘lgan $G(z)$ butun funksiyani ifodalashini ko‘rsatamiz.

Shu maqsadda ν qanday bo‘lishidan qat’iy nazar (1.17) ko‘paytma radiusi $|a_v|$ ga teng markazi 0 nuqtada bo‘lgan S doiraning ichida golomorf funksiyani ifodalashini isbotlaymiz.Tushunarliki, $|a_{v-1}| \leq |a_v|$ deb hisoblash yetarli. S_ν doiraning ichida nollarga ega bo‘lgan (5) dagi chekli sondagi $u_1, u_2, \cdots, u_{v-1}$ ko‘paytuvchilarni qaramasak qolgan ko‘paytuvchilar uchun quyidagiga ega bo‘lamiz:

$$u_v \cdot u_{v+1} \cdot \cdots = e^{-\sum_{n=v}^{\infty}\left[\frac{1}{n}\left(\frac{z}{a_n}\right)^n + \frac{1}{n+1}\left(\frac{z}{a_n}\right)^{n+1} + \cdots\right]}, \ |z| < |a_v|. \quad (1.18)$$

Agar biz $|z| < |a_v|$ bo‘lganda

$$\sum_{n=v}^{\infty}\left[\frac{1}{n}\left(\frac{z}{a_n}\right)^n + \frac{1}{n+1}\left(\frac{z}{a_n}\right)^{n+1} + \cdots\right] \quad (1.19)$$

qatorning golomorf funksiyaga yaqinlashishini ko‘rsatsak isbotlanishini talab etilgan tasdiqni isbotlagan bo‘lamiz. (1.19) -qatorning har bir hadi S_ν doiraning ichida golomorf funksiya. Agar ixtiyoriy yetarlicha kichik $\varepsilon > 0$ $soni\ uchun$ $|z| \leq (1 -$

$\varepsilon)|a_\nu|$ doirada tekis yaqinlashuvchi bo‘lsa (1.19)-qator ham Veyershtrassning 1-teoremasiga asosan S_ν doiraning ichida golomorf funksiya bo‘ladi.

Haqiqatan ham, $|z| \leq (1-\varepsilon)|a_\nu|$ bo‘lsa,

$$\left|\frac{1}{n}\left(\frac{z}{a_n}\right)^n + \frac{1}{n+1}\left(\frac{z}{a_n}\right)^{n+1} + \cdots\right| \leq \frac{1}{n}\left|\frac{z}{a_n}\right|^n + \frac{1}{n+1}\left|\frac{z}{a_n}\right|^{n+1} + \cdots$$

$$< \frac{1}{n}(1-\varepsilon)^n + \frac{1}{n+1}(1-\varepsilon)^{n+1} + \cdots < \frac{1}{n\varepsilon}(1-\varepsilon)^n.$$

Umumiy hadi $\frac{1}{n\varepsilon}(1-\varepsilon)^n$ bo‘lgan sonli qator yaqinlashuvchi bo‘lgani uchun $|z| \leq (1-\varepsilon)|a_\nu|$ bo‘lganda (1.19) qator tekis yaqinlashuvchi bo‘ladi.

Shunday qilib, biz (1.19)-qatorning S_ν doiraning ichida golomorf funksiyani ifodalashiga ishonch hosil qildik. Shuning uchun (1.17) ko‘paytma ham bu doiraning ichida golomorf funksiyani ifodalaydi; bu funksiya $z = a_1, a_2, \dots, a_{\nu-1}$da nolga aylanadi va qaralayotgan doiraning ichida shulardan boshqa nollarga ega emas. ν butun sonini yetarlicha katta qilib olish mumkin ekanligidan (1.17) ko‘paytmaning nollari berilgan $a_n$sonlaridan va faqat shu sonlardan iborat butun funksiyani ifodalashi kelib chiqadi.

Bu ko‘paytmadagi u_ν ko‘paytuvchilarga boshlang‘ich faktorlar deyiladi. u_ν boshlang‘ich faktorga $\left(1-\frac{z}{a_\nu}\right)$ chiziqli ko‘paytuvchidan tashqari ko‘rsatkichli faktor ham ega. Ana shu qo‘shimcha ko‘rsatkichli ko‘paytuvchilarning ishtirok etgani uchun (1.17) ko‘paytma yaqinlashuvchi bo‘ladi.

Hozirgacha berilgan (1.13) sonlar orasida nolga tenglari yo‘q deb faraz etgan edik.Agar $z = 0$ izlanayotgan butun funksiyaning λ tartibli noli bo‘lsa, hosil qilingan (1.17) ko‘paytmani z^λ ga ko‘paytirib olamiz.

Bu holda hosil qilingan

$$G(z) = z^\lambda \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{n-1}\left(\frac{z}{a_n}\right)^{n-1}}$$

formulaga Veyershtrass formulasi deyiladi.

Shunday qilib quyidagi natijani isbotladik.

1.1-natija (Veyershtrass formulasi). Agar $a_1, a_2, \ldots, a_n, \ldots$ 1.1-teorema shartni qanoatlantiruvchi kompleks sonlar ketma-ketligi bo‘lsa, u holda

$$G(z) = z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n-1}\left(\frac{z}{a_n}\right)^{n-1}}$$

Funksiya butun funksiya bo‘ladi va uning nollari faqat $0, a_1, a_2, \ldots, a_n, \ldots$ sonlaridan iborat bo‘ladi.

Bu formulani keltirib chiqarishda biz faqatgina berilgan (1.13) sonlar ketma-ketligini indeks n ning cheksiz o‘sishi bilan cheksizlikga intiladi deb faraz etdik. Ba’zi bir xususiy hollarda soddaroq ko‘rinishidagi boshlang‘ich faktorlardan foydalanish mumkin.

Umumiy hadi $\left|\frac{1}{a_n}\right|^{p+1}$ (bunda $r \geq 0 - butun\ son$) bo‘lgan qator yaqinlashuvchi bo‘lsin. Bu holda

$$u_v = \left(1 - \frac{z}{a_v}\right) e^{\frac{z}{a_v} + \frac{1}{2}\left(\frac{z}{a_v}\right)^2 + \frac{1}{3}\left(\frac{z}{a_v}\right)^3 + \cdots + \frac{1}{p}\left(\frac{z}{a_v}\right)^p} \qquad (1.15')$$

deb olish mumkin. Haqiqatan ham yuqorida qarab chiqilgan tahlilimizga asosan masala

$$\sum_{n=v}^{\infty} \left[\frac{1}{p+1}\left(\frac{z}{a_n}\right)^{p+1} + \frac{1}{p+2}\left(\frac{z}{a_n}\right)^{p+2} + \cdots\right] \qquad (1.19')$$

qatorning $|z| \leq (1-\varepsilon)|a_v|$da tekis yaqinlashuvchi ekanligini isbotlashga olib kelinadi. Bu yerda

$$\left|\frac{1}{p+1}\left(\frac{z}{a_n}\right)^{p+1} + \frac{1}{p+2}\left(\frac{z}{a_n}\right)^{p+2} + \cdots\right| \leq \frac{1}{p+1}\left|\frac{z}{a_n}\right|^{p+1} + \frac{1}{p+2}\left|\frac{z}{a_n}\right|^{p+2} + \cdots$$

$$< \frac{1}{p+1} \frac{\left|\frac{z}{a_n}\right|^{p+1}}{1 - \left|\frac{z}{a_n}\right|} < \frac{1}{p+1} \frac{(1-\varepsilon)^{p+1}|a_n|^{p+1}}{\varepsilon} \cdot \left|\frac{1}{a_n}\right|^{p+1}$$

bo‘lib umumiy hadi $\frac{1}{p+1}\frac{(1-\varepsilon)^{p+1}|a_n|^{p+1}}{\varepsilon} \cdot \left|\frac{1}{a_n}\right|^{p+1}$ bo‘lgan sonli qator shartli yaqinlashuvchi. Demak, (1.15’) qator ham yaqinlashuvchi. Bu yerda quyidagi 1.2-

natija o‘rinli.

1.2-natija. Agar $a_1, a_2, \ldots, a_n, \ldots$ sonlar ketma-ketligi 1.1-teoremaning shartlarini qanoatlantirsa va shunday butun $c\ \ p \geq 0$ soni mavjud bo‘lib

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{c\ \ p+1}}$$

Qator yaqinlashuvchi bo‘lsa, u holda

$$G_1(z) = z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{c\ \ p}\left(\frac{z}{a_n}\right)^{n-1}}$$

funksiya 1.1-teoremani qanoatlantiruvchi butun funksiya bo‘ladi.

Butun funksiyani cheksiz ko‘paytmalar ko‘rinishida ifodalash.

Biz bunda noldingi ma’ruzamizda (1.14)- shartni qanoatlantiruvchi berilgan (1.13) sonlar ketma-ketligi nollari bo‘lgan Veyershtrass formulasi bilan ifodalanadigan $G(z)$butun funksiya mavjud ekanligini ko‘rsatdik. Aksincha agar cheksiz ko‘p nollarga ega bo‘lgan butun funksiya $G_1(z)$ berilgan bo‘lsa, u holda ma’lumki, bu nollar limit nuqtaga ega bo‘lmasligi kerak, ya’ni ularning modulining o‘sib borish tartibida $a_1, a_2, \ldots, a_n, \ldots$sonlar ketma-ketligi ko‘rinishida joylashtirish mumkin va u n ning o‘sishi bilan cheksizlikga intiladi. Veyershtrass formulasi bo‘yicha nollari ana shu sonlardan iborat bo‘lgan $G(z)$butun funksiyani qurib

$$\varphi(z) = \frac{G_1(z)}{G(z)} \qquad (1.20)$$

nisbatning ham butun funksiyani ifodalashini ko‘ramiz. Bu funksiya nolga aylanmaydi va $\varphi(a_n) = \lim_{z \to a_n} \varphi(z)$. Bu shartlarda$\frac{\varphi'(z)}{\varphi(z)}$ifoda ham butun funksiya bo‘ladi. Shuning uchun ham $\varphi(z) = e^{H(z)}$ deb yoza olamiz, bu yerda $H(z)$ biror butun funksiya. Bulardan foydalanib (1.20) ni

$$G_1(z) = e^{H(z)} \cdot G(z) \qquad (1.20')$$

yoki

$$G_1(z) = e^{H(z)} * z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{a_n}\right) e^{\frac{\lambda}{a_n} + \frac{1}{2}\left(\frac{\lambda}{a_n}\right)^2 + \cdots + \frac{1}{n-1}\left(\frac{\lambda}{a_n}\right)^{n-1}}$$

Amaliyotda $G_1(z)$ berilgan funksiya uchun $H(z)$ funksiyani aniqlash ancha qiyin kechadi. Misol uchun $G_1(z) = sinz$ bo‘lsa, uning nollari $z = n\pi,\ n \in Z.$ Shuning uchun ham

$$sinz = e^{H(z)} \cdot z \prod\nolimits_n' \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}}$$

Bu yerda ko‘paytmaning musbat va manfiy butun qiymatlari bo‘yicha olinadi. Biz bu yerda dastlabki faktorni soddaroq ko‘rinishda olishimiz mumkin. Chunki

$$\sum \left|\frac{1}{a_n}\right|^p = \sum \left|\frac{1}{n\pi}\right|^p$$

qator $p = 2$ da yaqinlashuvchi.

$sinz$ uchun olingan formulani yanada soddalatirish uchun nollar $\pi, -\pi, 2\pi, -2\pi, \dots, n\pi, -n\pi, \dots$ ga mos ko‘paytuvchilarni ikkitadan birlashtirib olamiz, u holda

$$sinz = e^{H(z)} \cdot z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

$H(z)$- butun funksiyasini quyidagicha aniqlashimiz mumkin. Yuqoridagi tenglikning ikkala tomonini logarifmlab quyidagiga ega bo‘lamiz:

$$lnsinz = lne^{H(z)} + lnz + \sum_{n=1}^{\infty} ln\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Bundan

$$(lnsinz)' = ctgz = H'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}.$$

Bu formulani qoldiqlar nazariyasida isbotlangan.

$$ctgz = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

Formula bilan taqqoslasak, $H'(z) \equiv 0$ bo‘lishi kerak degan xulosaga kelamiz. Bundan esa $H(z) = const$ ekanligi kelib chiqadi. Shunday qilib

$$sinz = C \cdot z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Endi C ning qiymatini aniqlash uchun

$$\frac{sinz}{z} = C \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

ko‘rinishda yozib olib $z \to 0$limitga o‘tsak, $C = 1$kelib chiqadi va biz

$$sinz = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \tag{1.21}$$

ga ega bo‘lamiz.

I.3-§. Chekli tartibli butun funksiyalar.

1.3-ta’rif. $G(s) -$ butun funksiyava $M(r) = M_G(r) = \max\limits_{|s|=r} |G(s)|$ bo‘lsin.

Agarda

$$M(r) < e^{ra}, \quad r > r_0(a) > 0 \tag{1.22}$$

Shartni qanoatlantiruvchi $a > 0$ soni mavjud bo‘lsa, u holda $G(s)$ ga chekli tartibli butun funksiya deyiladi. Bu holda $\alpha = \inf a$ ga $G(s)$ tartibi deyiladi.Agarda (1.22) ni qanoatlantiruvchi chekli a mavjud bo‘lmasa $G(s)$ ning tartibi cheksizga teng deyiladi.

1.4-ta’rif. Agar $s_1, s_2, \ldots, s_n, \ldots$ lar

$$0 < |s_1| \le |s_2| \le \ldots \le |s_n| \le \ldots \tag{1.23}$$

Shartni qanoatlantiruvchi kompleks sonlar ketma-ketligi bo‘lib

$$\sum_{n=1}^{\infty} \frac{1}{|s_n|^b} < \infty \tag{1.24}$$

Tengsizlikni qanoatlantiruvchi $b > 0$ soni mavjud bo‘lsa, u holda (1.24)-ketma-ketlik *chekli yaqinlashish tartibiga ega* deyiladi. $\inf b = \beta$ soniga (1.23) ning *yaqinlashish ko‘rstkichi* deyiladi. Agarda (1.24) ni qanoatlantiruvchi $b > 0$ soni mavjud bo‘lmasa (1.23) ning yaqinlashish darajasi *cheksizga teng* deyiladi.

$\xi(s)$ funksiyaning nollarini tekshirishda bizga butun funksiyalar nazariyasiga tegishli yana quyidagi teorema kerak bo‘ladi.

1.4-teorema. Agar $G(s)$ chekli α tartibli butun funksiyaga $G(0)\neq 0$ bo‘lib $\{s_n\}$ $G(s)$ barcha nollari ketma-ketligi

$$0<|s_i|\leq|s_2|\leq\ldots\leq|s_n|\leq\ldots$$

Shartni qanoatlantirsa, u holda $\{s_n\}$ chekli $\beta\leq\alpha$ yaqinlashish ko‘rsatkichiga ega va

$$G(s)=e^{g(s)}\prod_{n=1}^{\infty}\left(1-\frac{s}{s_n}\right)\ e^{\frac{s}{s_n}+\frac{1}{2}\left(\frac{s}{s_n}\right)^2+\cdots+\frac{1}{\rho}\left(\frac{s}{s_n}\right)^{\rho}},$$

buyerda $\rho\geq 0$

$$\sum_{n=1}^{\infty}\frac{1}{|s_n|^{\rho+1}}<\infty$$

Shartni qanoatlantiruvchi eng kichik butun son, $g(s)$ esa $g\leq\alpha$ darajali ko‘phad va $\alpha=\max(g;\ \beta)$.

Agarda bulardan tashqari har qanday $c>0$ uchun shunday bir cheksiz ketma-ketlik mavjud bo‘lsaki

$$r_1, r_2, \ldots, r_n, \ldots;\quad r_n\to\infty$$

$$\max|G(s)|>e^{c\ r_n},\ \ |s|=r_n,\ \ n=1,\ 2,\ 3,\ldots$$

bajarilsa u holda $\alpha=\beta$ bo‘ladi va

$$\sum_{n=1}^{\infty}\frac{1}{|s_n|^{\beta}}$$

qator uzoqlashuvchi bo‘ladi.

Birinchi bob yuzasidan xulosa

Ushbu dissertatsiyaning birinchi bobida cheksiz ko‘paytmalar funksiyalarni cheksiz qatorlardan tashqari cheksiz ko‘paytmalar yordamida ifodalash muhim ahamiyatga ekanligi yoritilgan. Yaqinlashuvchi va uzoqlashuvchi cheksiz ko‘paytmalar, cheksiz ko‘paytma yaqinlashishining zaruriy va yetarli shartlari yoritib berilgan. Cheksiz ko‘paytmalarning yaqinlashuvchi bo‘lishini tengsizlik yordamida ifodalangan Cheksiz ko‘paytmalarga doir misollar yechimi ko‘rsatilgan.

Chekli tartibli butun funksiyalar, Veyershtrass formulasi haqida ma'lumot berilgan. Chekli tartibli butun funksiyalarni cheksiz ko‘paytmalar yordamida ifodalash ko‘rsatilgan, misollar keltirilgan.

II-BOB. $\psi(x)$ FUNKSIYASI UCHUN ANIQ FORMULA

II.1-§. $\zeta(s)$-funksiyaning logarifmik hosilasini nollari bo‘yicha qatorga yoyish

Agar $\sigma > 1$ bo‘lsa, $\zeta(s) \neq 0$ va agar $\sigma < 0$ bo‘lsa, $\zeta(s)$ funksiyasi trivial bo‘lmagan nollarga ega emas ekanligi isbotlangan. Tekislikning qolgan qismi, ya’ni $0 \leq \sigma \leq 1$ga kritik yo‘lak (polosa) deb ataladi. Bulardan tashqari Riman$\zeta(s\ )$ to‘g‘risida bir necha gipotezalarni ilgari suradi. Ulardan biri $\zeta(s)$ ning barcha trivial bo‘lmagan nollari $\sigma = \frac{1}{2}$ kritik to‘g‘ri chiziqda yotadi, degan gipotezasi hozirgacha to‘la isbotlangan emas. Bu gipotezaga hozirda Riman gipotezasi deb yuritiladi[7].

1914-yilda G.Xardi $\sigma = \frac{1}{2}$ to‘g‘ri chiziqda $\zeta(s)$ ning cheksiz ko‘p nollarining yotishini isbotladi. 1942-yilda A. Selberg esa bu nollarning $\zeta(s)$ ning barcha nollari orasida musbat zichlikka ega ekanligini isbotladi.

Valle-Pussen va Adamarlar 1898-yilda bir-biriga bog‘liq bo‘lmagan holda $s = 1$ da $\zeta(s) \neq 0$ ekanligini isbotladilar. Aniqroq qilib aytganda Valle-Pussen, agar

$$\sigma > 1 - \frac{c_1}{lnt}, \quad t \geq 2 \tag{2.1}$$

bo‘lsa, u holda $\zeta(s) \neq 0$ ekanligini ko‘rsatgan. Bu yerda c_1 −qandaydir musbat o‘zgarmas son.

1948-yilda A. Selberg va P. Erdyoshlar bu natijaning elementar isbotini berdilar. Shundan keyin N. Chudakov agar

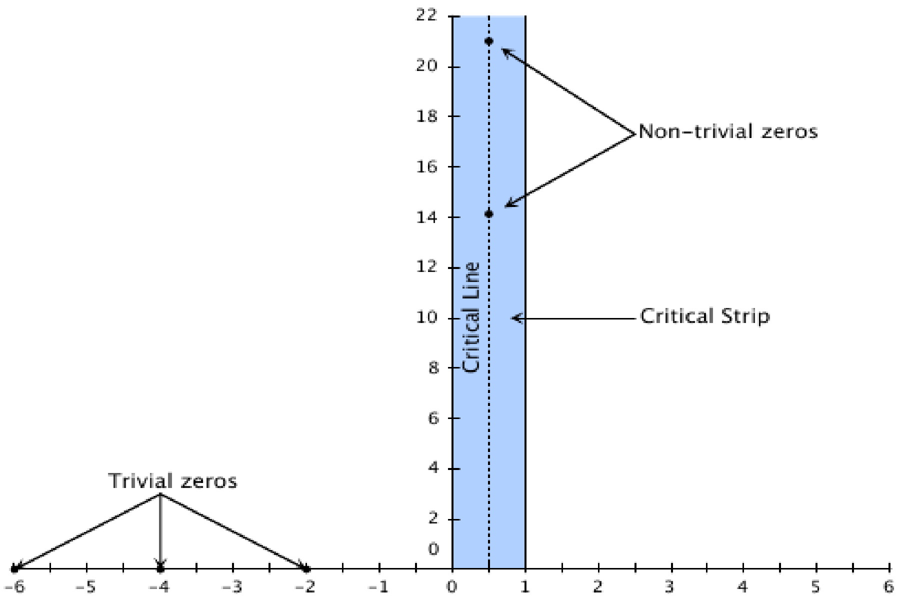
$$\sigma > 1 - \frac{c_2}{\left(ln^{\frac{3}{4}}t\right)(lnlnt)^{\frac{3}{4}}}$$

bo‘lsa, $\zeta(s) \neq 0$ ekanligini isbot qildi. 1958 yilda I. M. Vinogradov va N.M.Korobovlar agar

$$\sigma > 1 - \frac{C(\alpha)}{(lnt)^{\alpha}}, \quad \alpha > \frac{2}{3}$$

bo‘lsa, u holda $\zeta(s) \neq 0$ ekanligini ko‘rsatishdi.

Hozirgi vaqtda $\zeta(s)$ ning eng kichik ordinatali noli $\beta_1 = \frac{1}{2} + i14{,}134725$ ekanligi isbotlangan [9]. $\zeta(s)$ ning nollari haqiqiy o‘qga nisbatan simmetrik joylashgani uchun $\bar{\beta}_1 = \frac{1}{2} - i14{,}134725$ ham $\zeta(s)$ ning noli bo‘ladi. Demak, $0 \leq \sigma \leq 1$, $-14{,}134725 < t < 14{,}134725$ to‘g‘ri to‘rtburchakning ichida $\zeta(s)$ ning nollari yo‘q deya olamiz. Shuningdek, $\zeta(s)$ ning ikkinchi va uchinchi trivial bo‘lmagan nollari $\beta_2 = \frac{1}{2} + i21{,}022$; $\bar{\beta}_2 = \frac{1}{2} - i22{,}022$; $\beta_3 = \frac{1}{2} + i25{,}011$, $\bar{\beta}_3 = \frac{1}{2} - i25{,}011$ ekanligi ma’lum [10] (1-shaklga qarang).



2.1-shakl

Kompyuter yordamida ordinatasi $0 < t < 33 \cdot 10^9$ shartni qanoatlantiruvchi barcha nollari $\sigma =$ to‘g‘ri chiziq ustida yotishi isbotlangan. Biz quyida $\zeta(s)$ nollari qaysi olimlar tomonidan kashf etilgani haqidagi ma’lumotlar jadvalini keltiramiz([7],[10]).

Yillar	Nollari soni	Kim tomonidan topilgani
1859 (taxmin qilgan.)	1	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1041	E. C. Titchmarsh
1953	1104	A. M. Turing
1956	15000	D. H. Lehmer

1956	25000	D. H. Lehmer
1958	35337	N. A. Meller
1966	250000	R. S. Lehman
1968	3500000	J. B. Rosser va boshqalar
1977	40000000	R. P. Brent
1979	81000001	R. P. Brent
1982	200000001	R. P. Brent va boshqalar
1983	300000001	J. van de Lune, H. J. J. te Riele
1986	1500000001	J. van de Lune va boshqalar
2001	10000000000	J. van de Lune
2004	900000000000	S. Wedeniwski
2004	10000000000000	X. Gourdon

2.1- jadval

Shuning uchun ham bu sohadagi izlanishlar aktual hisoblanadi.

I. Allakovning [11] da

$$\sum_{n=1}^{\infty}\frac{1}{1+(T-\gamma_n)^2}\leq c_3 lnT,\ \ T\geq T_0\geq 3 \qquad (2.2)$$

bahodagi c_3 ning qiymati aniqlashtirilib sonli baho olinganini ta’kidlab o‘tamiz. Aniqroq qilib aytganda ning T_0 orqali ifodaga keltirib chiqarilib $T_0=3$ da hisoblangan va $c_3=5{,}4695$ natija olingan, ya’ni quyidagi teorema isbotlagan[12].

2.1-teorema. Agar $q_n=\beta_n+i\gamma_n, n=1,2,3,\dots$ lar $\zeta(s)$ ning barcha trivial bo‘lmagan nollari bo‘lsa, u holda

$$\sum_{n=1}^{\infty}\frac{1}{1+(T-\gamma_n)^2}\leq c_3 lnT,\ \ T\geq T_0\geq 3 \qquad (2.3)$$

munosabat o‘rinli. Bu yerda

$$c_3=\frac{4}{(T_0^2+1)lnT_0}+\frac{1{,}5834+ln2\pi}{lnT_0}+1+\frac{\gamma}{lnT_0}+\frac{3}{2T_0lnT_0}+\frac{1}{12T_0^2lnT_0}\leq 5{,}4695$$

va $\gamma=0{,}5772\dots-$ Eyler doimiysi.

Agarda biz bu bahoda $T_0=14$ (bunday deb olish mumkin ekanligi yuqorida isbotlandi) deb olsak $c_3\leq 2{,}28$ bahoga ega bo‘lamiz.

Bunday baholar (2.1) ko‘rinishdagi baholarda qatnashuvchi c_1 doimiylarning son qiymatlarini aniqlashda muhim hisoblanadi. Isbotlangan tasdiqdan ushbu natijalar kelib chiqadi.

2.1-natija. $\zeta(s)$ ning $T \leq |Imq_n| \leq T+1$ shartni qanoatlantiruvchi nollari soni $2{,}28lnT$ dan ko‘p emas.

2.2-natija. Agar $T \geq T_0 \geq 3$ bo‘lsa,

$$\sum_{|T-\gamma_n|>1} \frac{1}{(T-\gamma_n)^2} \leq c_4 lnT, \tag{2.4}$$

$$c_4 = 5\left(\frac{1}{(T_0^2+1)lnT_0} + \frac{0{,}22305 + ln2\pi}{lnT_0} + 1 + \frac{3}{2T_0 lnT_0} + \frac{1}{12T_0^2 lnT_0}\right) \leq 7{,}61.$$

Olingan sonli natijalar [11] dagi natijalarning yaxshilangani hisoblanadi.

Dzeta funksiyaning nollari haqidagi ba’zi teoremalar

$\zeta(s)$ ning funksional tenglamasidan ko‘rinadiki, $s=-2,-4,\ldots,-2n,\ldots$ nuqtalarda $\zeta(s)=0$ bo‘ladi, chunki bu nuqtalarda $\Gamma^{-1}\left(\frac{s}{2}\right)=0$. Haqiqatan ham ta’rifga asosan

$$\Gamma^{-1}(s) = s e^{\gamma s} \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}, \tag{2.5}$$

bu yerda γ –Eyler doimiysi.

(2.5) ning o‘ng tomoni yuqorida ko‘rsatilgan qiymatlarda nolga aylanadi. S=0 bo‘lganda $\zeta(s) \neq 0$, chunki bu holda $\Gamma^{-1}\left(\frac{s}{2}\right)$ ning noli $\zeta(1-s)$ ning qutbi bilan qisqarib ketadi. $\zeta(s)$ ning bunollariga uning trivial nollari deyiladi. $\zeta(s)$ funksiyasi bulardan tashqari $0 \leq \operatorname{Re} s \leq 1$yo‘lakda (kritik yo‘lakda) yotuvchi trivial bo‘lmagan cheksiz ko‘p nollarga ham ega.

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}} \quad \tilde{A}\left(\frac{s}{2}\right)\zeta(s) \tag{2.6}$$

deb olsak (1.1-§ga qarang) ushbu teorema o‘rinli:

2.2-teorema. $\xi(s)$ funksiyasi birinchi tartibli butun funksiya bo‘lib $0 \leq \operatorname{Re} q_n \leq 1$ shartni qanoatlantiruvchi cheksiz ko‘p nollar q_n ga ega;

$\sum_n |q_n|^{-1}$ –qator uzoqlashuvchi;

$\sum_n |q_n|^{-1-\varepsilon}$ -qator esa ixtiyoriy $\varepsilon > 0$ uchun yaqinlashuvchi bo‘ladi;

$\xi(s)$ ning nollari $\zeta(s)$ ning trivial bo‘lmagan nollaridan iboratdir.

Isboti. Agar $\operatorname{Re} s > 1$ bo‘lsa $\xi(s)$ funksiya va demak $\zeta(s)$ ham nollarga ega emas (1.1.1-lemmaning natijasiga qarang). 1.1.1-teoremadan $\operatorname{Re} s < 0$ bo‘lganda ham $\zeta(s) \neq 0$ ekanligi kelib chiqadi. $\zeta(0) = \zeta(1) \neq 0,$ bo‘lgani uchun $\xi(s)$ ning nollari faqat $\zeta(s)$ ning trivial bo‘lmagan nollaridan iborat bo‘ladi[13].

Endi $\xi(s)$ ning tartibini aniqlaylik. Buning uchun $\xi(s)$ ni $|s| \to \infty$ da baholaymiz. 1.1.2-lemmadan $\operatorname{Re} s \geq \frac{1}{2}$ bo‘lganda quyidagini hosil qilamiz:

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^{1/2}} + \frac{N^{1/2}}{|s-1|} + \frac{1}{N^{1/2}} + |s| \left| \int_{N^2}^{\infty} \frac{du}{u^{3/2}} \right|,$$

ya’ni $\zeta(s) = 0\left(|s|\right)$. [7] dagi 2.1-teoremaning 1-natijasiga asosan

$$|\Gamma(s)| \leq e^{c|s|\ln|s|}$$

bo‘lganidan $\xi(s)$ ning tartibi birdan katta emas degan xulosaga kelamiz. Lekin da $|s| \to \infty$ da $\ln \Gamma(s) \sim s \ln s$ bo‘lgani uchun ham $\xi(s)$ ning tartib 1 ga teng. [8] dagi 1.3-teoremadan $\sum_n |q_n|^{-1}$ qatorning uzoqlashuvchi ekanligi (bu yerda q_n lar $\xi(s)$ ning nollari) kelib chiqadi va shuning uchun ham $\xi(s)$ cheksiz ko‘p nollarga ega bo‘ladi. U holda

$$\sum_n |q_n|^{-1-\varepsilon}$$

qator esa ixtiyoriy $\varepsilon > 0$ soni uchun yaqinlashuvchi bo‘ladi. Teorema to‘liq isbot bo‘ldi.

2.3-natija. Ushbu formula o‘rinli

$$\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{q_n}\right) e^{\frac{s}{q_n}}. \tag{2.7}$$

2.4-natija. $\zeta(s)$ ning trivial bo‘lmagan nollari $\operatorname{Re}s=\frac{1}{2}$ va $\operatorname{Im}s=0$ to‘g‘ri chiziqlarga nisbatan simmetrik joylashgan.

Bundan keyin biz $\zeta(s)$ ning trivial bo‘lmagan nollarini ularning mavhum qismlarining absolyut qiymatlarining o‘sib borish tartibida nomerlaymiz. Agar mavjud qismlarining absolyut qiymatlari teng bo‘lsa, ularni ixtiyoriy tartibda nomerlab yozaveramiz.

2.3-teorema. Ushbu tenglik o‘rinli

$$\frac{\zeta'(s)}{\zeta(s)}=-\frac{1}{s-1}+\sum_{n=1}^{\infty}\left(\frac{1}{s-q_n}+\frac{1}{q_n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)+B_0,$$

bu yerda q_n lar $\zeta(s)$ ning barcha trivial bo‘lmagan nollari, B_0 – parametrga bog‘liq bo‘lmas (absolyut) doimiy son.

Isboti. (2.7) ning ikkala tomonini logarifmlaymiz, u holda

$$\ln\xi(s)=A+Bs+\sum_{n=1}^{\infty}\left(\ln\left(1-\frac{s}{q_n}\right)+\frac{s}{q_n}\right).$$

buni deferensiallab

$$\frac{\xi'(s)}{\xi(s)}=B+\sum_{n=1}^{\infty}\left(\frac{1}{s-q_n}+\frac{1}{q_n}\right) \qquad (2.8)$$

ni hosil qilamiz. (2.6) asosan

$$\ln\ \xi(s)=\ln\left(\frac{s}{2}\ \tilde{A}\left(\frac{s}{2}\right)\right)+\ln(s-1)-\frac{s}{2}\ln\pi+\ln\zeta\ (s)$$

yoki

$$\ln\ \xi(s)=\ln\left(\tilde{A}\left(\frac{s}{2}+1\right)\right)+\ln(s-1)-\frac{s}{2}\ \ln\pi+\ln\zeta(s)\ .$$

Oxirgi tenglamani differensiallasak

$$\frac{\xi'(s)}{\xi(s)}=\frac{1}{2}\ \frac{\tilde{A}'\left(\frac{s}{2}+1\right)}{\tilde{A}\left(\frac{s}{2}+1\right)}+\frac{1}{s-1}-\frac{1}{2}\ \ \ln\pi+\frac{\zeta'(s)}{\zeta(s)}.$$

Buni (2.8) ga olib borib qo‘ysak

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\ln\pi - \frac{1}{2}\frac{\tilde{A}'\left(\frac{s}{2}+1\right)}{\tilde{A}\left(\frac{s}{2}+1\right)} + \sum_{n=1}^{\infty}\left(\frac{1}{s-q_n} + \frac{1}{q_n}\right) \tag{2.9}$$

ni hosil qilamiz. $\Gamma(s)$ ning ta’rifidan

$$\frac{1}{\frac{s}{2}\,\Gamma\left(\frac{s}{2}\right)} = e^{\gamma\frac{s}{2}}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}$$

ga ega bo‘lamiz. Buni logarifmlab

$$-\ln\,\Gamma\left(\frac{s}{2}+1\right) = \frac{\gamma s}{2} + \sum_{n=1}^{\infty}\left(\ln\left(1+\frac{s}{2n}\right) - \frac{s}{2n}\right)$$

yoki buni differensiallasak

$$-\frac{\tilde{A}'\left(\frac{s}{2}+1\right)}{2\,\tilde{A}\left(\frac{s}{2}+1\right)} = \frac{\gamma}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{2n+s} - \frac{1}{2n}\right) \tag{2.10}$$

kelib chiqadi. (2.10) ni (2.9) ga olib borib qo‘ysak isbotlanishi talab etilgan tenglik kelib chiqadi.

2.3-teorema. Agar $q_n = \beta_n + i\gamma_n,\ n = 1, 2, 3, \ldots$lar $\zeta(s)$ ning barcha trivial bo‘lmagan nollari va $T \geq 3$ bo‘lsa, u holda

$$\sum_{n=1}^{\infty}\frac{1}{1+(T-\gamma_n)^2} \leq C_3 \ln T,$$

bu yerda $T \geq T_0 \geq 3$va

$$C_3 = \frac{4}{\left(T_0^2+1\right)\ln T_0} + \frac{1{,}5834 + \ln 2\pi}{\ln T_0} + 1 + \frac{\gamma}{\ln T_0} + \frac{3}{2T_0\ln T_0} +$$
$$+ \frac{1}{12T_0^2\ln T_0} \leq 5{,}4695.$$

Isboti. $s = 2 + iT$ deb olamiz. U holda

$$\left|\sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right)\right| \leq \sum_{n\leq T}\left(\frac{1}{2n} + \frac{1}{2n}\right) + \sum_{n>T}\frac{|s|}{4n^2} = \sum_{n\leq T}\frac{1}{n} + \frac{|s|}{4}\sum_{n>T}\frac{1}{n^2}.$$

[8] da (1.8) formulaga asosan

$$\sum_{n\le T}\frac{1}{n}<\ln T+\gamma+\frac{1}{2T}+\frac{1}{12T^2}\ \text{va}$$

$$\sum_{n>T}\frac{1}{n^2}<\int_T^{\infty}\eta^{-2}\,d\eta=-\eta^{-1}\Big|_T^{\infty}=\frac{1}{T}.$$

Shuning uchun ham

$$\left|\sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)\right|<\ln T+\gamma+\frac{3}{2T}+\frac{1}{12T^2}.$$

2.3-teoremadan

$$-\mathrm{Re}\frac{\zeta'(s)}{\zeta(s)}=\mathrm{Re}\left(\frac{1}{s-1}-B_0-\sum_{n=1}^{\infty}\left(\frac{1}{s+n}-\frac{1}{2n}\right)\right)-$$
$$-\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{1}{s-q_n}-\frac{1}{q_n}\right)\le c_1\ln T-\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{1}{s-q_n}+\frac{1}{q_n}\right), \tag{2.11}$$

bu yerda $T\ge T_0\ge 3$ va

$$C_1=\frac{1}{\left(T_0^2+1\right)\ln T_0}+\frac{-1+\ln 2\pi}{\ln T_0}+1+\frac{\gamma}{\ln T_0}+\frac{3}{2T_0\ln T_0}+\frac{1}{12\,T_0^2\ln T_0}.$$

Bunda biz

$$\mathrm{Re}\frac{1}{s-1}=\mathrm{Re}\frac{1}{1+iT}=\frac{1}{T^2+1}\le\frac{1}{T_0^2+1},\quad T\ge T_0(\ge 3)$$

$$B=-\frac{1}{2}\gamma-1+\frac{1}{2}\ln 4\pi$$

$$B_0=B+\frac{1}{2}\ln\pi+\frac{\gamma}{2}=-1+\ln 2\pi$$

(bu[6]ning 12-§ dagi (2.10) formula) ekanliklaridan foydalandik. Eyler ayniyatiga ko‘ra

$$\zeta(s)=\prod_p\left(1-p^{-s}\right)^{-1}.$$

Bundan

$$\ln\zeta(s)=-\sum_p\ln\left(1-p^{-s}\right).$$

$$\log\left(1-p^{-s}\right)^{-1}=\ln\frac{1}{1-p^{-s}}=\frac{1}{p^{s}}+\frac{1}{2p^{2s}}+\frac{1}{3p^{3s}}+\cdots$$

(bu bizga ma’lum bo‘lgan

$$\ln\frac{1}{1-u}=u+\frac{u^{2}}{2}+\frac{u^{3}}{3}+\ldots,\qquad 0<u<1$$

formuladan $u=p^{-s}$ deb olsak kelib chiqadi).

Shunday qilib

$$\ln\zeta(s)=\sum_{p}\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}.$$

Oxirgi tenglikning ikkala tomonini differensiallasak

$$\frac{\zeta'(s)}{\zeta(s)}=\sum_{p}\sum_{m=1}^{\infty}\frac{\ln p}{p^{ms}}=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}},$$

bu yerda $\Lambda(n)$ Mangoldt funksiya bo‘lib

$$\Lambda(n)=\begin{cases}\ln p\,, & n=p^{m}\\ 0\,, & n\neq p\end{cases}$$

tenglik bilan aniqlanadi.

Demak

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right|=\left|\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2+iT}}\right|\leq\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2}}=\sum_{n\leq 41}\frac{\Lambda(n)}{n^{2}}+$$

$$+\sum_{n>41}\frac{\Lambda(n)}{n^{2}}<0,53088+\int_{41}^{\infty}\frac{\ln x}{x}dx=0,64585$$

[9] dagi (1.2.4)-formulaga qarang).

Bularni e’tiborga olsak (2.10) dan

$$\operatorname{Re}\sum_{n=1}^{\infty}\left(\frac{1}{s-q_{n}}+\frac{1}{q_{n}}\right)\leq C_{2}\ln T,$$

bunda

$$C_{2}=C_{1}+0{,}64585\,\left(\ln T_{0}\right)^{-1},\quad\left(T\geq T_{0}\geq 3\right).$$

Agar bu yerda

$$\operatorname{Re}\frac{1}{s-q_n}=\operatorname{Re}\frac{1}{(2-\beta_n)+i(T-\gamma_n)}=\frac{2-\beta_n}{(2-\beta_n)^2+(T-\gamma_n)^2}\geq$$
$$\geq\frac{1}{4+(T-\gamma_n)^2}>\frac{1/4}{1+(T-\gamma)^2} \qquad (2.12)$$

va $\operatorname{Re}\frac{1}{q_n}=\frac{\beta_n}{\beta_n^2+\gamma_n^2}\geq 0$

ekanliklarini e'tiborga olsak

$$\sum_{n=1}^{\infty}\frac{1}{1+(T-\gamma_n)^2}\leq C_3\ln T,\quad C_3=4C_2.$$

isbotlangan teoremadan ushbu natijalar kelib chiqadi.

2.6-natija. $\zeta(s)$ ning $T\leq|\operatorname{Im}q_n|\leq T+1$ shartni qanoatlantiruvchi nollari soni $C_3\ln T$ dan ko‘p emas.

2.7-natija. Agar $T\geq T_0(\geq 3)$ bo‘lsa

$$\sum_{|T-\gamma_n|>1}\frac{1}{|T-\gamma_n|^2}\leq C_4\ln T,\quad C_4=5\cdot C_2\leq 6{,}8369\,.$$

Isboti. Tushunarliki

$$\sum_{n=1}^{\infty}\frac{1}{4+(T-\gamma_n)^2}\geq\sum_{|T-\gamma_n|>1}\frac{1}{5\,(T-\gamma_n)^2}\,.$$

Bu yerdan va 2.3-teoremadan isbotlanishi talab etilayotgan natija kelib chiqadi[13]. Shuni ham ta’kidlash kerakki, 2.3-teoremaga asosan $C_4=5C_3$ bo‘lishi kerak, lekin (2.12) dan qaralayotgan holda $C_4=5C_2$ deb olish mumkin ekanligi kelib chiqadi[7].

II.2-§. $N(T)$ funksiya uchun formulaning qoldiq hadiga kiruvchi o‘zgarmasning qiymatini aniqlash

Bu paragrafda biz Davenport ([9], 8-paragraf (1) ga qarang) da keltirilgan asimtotik formulani isbotlaymiz. $\xi(s)$ funksiyaning $0<\sigma<1,\ 0<t<T$ to‘gri to‘rtburchakda joylashgan, nollar soni *N(T)* ga teng bo‘lgan formulani Riman tavsiya qilgan va Mongolt isbotlab bergan edi.

$\zeta(s)$ funksiyadan ko‘ra $\xi(s)$ funksiya bilan ishlash osonroqdir. Chunki u oddiy funksional tenglamani qanoatlantiradi. Oson bo‘lishi uchun T nolning ordinatasidan farqli bo‘lsin.

Bu holda

$$2\pi N(T) = \Delta_R arg\xi(s)$$

Bunda, R s-tekislikda, uchlari 2, $2+iT$, $-1+iT$, -1 nuqtalarda joylashgan to‘g‘ri to‘rburchak (uning musbat yo‘nalishi qaraladi) S to‘rtburchak asosi bo‘ylab harakatlanganda arg$\xi(s)$ ortirmaga ega emas, chunki $\xi(s)$ funksiya haqiqiy bo‘lib nolga aylanmaydi.

U holda

$$\xi(\sigma+it) = \xi(1-\sigma-it) = \overline{\xi(1-\sigma+\imath t)}$$

bo‘lganligi sababli $arg\xi(s)$ s $\frac{1}{2}+iT$ dan $-1+iT$ gacha va keyinchalik -1 ga o`zgarganda, s 2 dan $2+iT$ gacha va keyinchalik $\frac{1}{2}+iT$ gacha o`zgarganda bir xil orttirma qabul qiladi.

Demak

$$\pi N(T) = \Delta_L arg\xi(s)$$

bunda L bilan $2, 2+iT$ va $\frac{1}{2}+iT$ nuqtalarni tutashtiruvchi siniq chiziq belgilangan.

ξ(s) ni aniqlaydigan tenglikni quyidagicha yozib olish mumkin[9].

$$\xi(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s+1\right)\zeta(s)$$

Bu holda

$$\Delta_L arg(s-1) = arg\left(iT-\frac{1}{2}\right) = \frac{1}{2}\pi + O(T^{-1})$$

$$\Delta_L arg\pi^{-\frac{1}{2}s} = \Delta_L\left(-\frac{1}{2}t\log\pi\right) = -\frac{1}{2}T\log\pi.$$

Bundan tashqari Stirling ([9], 10-paragrafdagi (1)ga qarang) formulasiga asosan

$$\Delta_L\, arg\,\Gamma\left(\frac{1}{2}s+1\right) = Im\,log\ \Gamma\left(\frac{1}{2}iT+\frac{5}{4}\right) =$$

$$= Im\left[\left(\frac{1}{2}iT+\frac{3}{4}\right)\log\left(\frac{1}{2}iT+\frac{5}{4}\right)-\frac{1}{2}iT-\frac{5}{4}+\frac{1}{2}log\,2\pi+O(T^{-1})\right] =$$

$$= \frac{1}{2}T\,log\frac{1}{2}T-\frac{1}{2}T+\frac{3}{8}\pi+O(T^{-1})$$

Demak

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+S(T)+\ O(T^{-1}) \qquad (2.13)$$

Bunda

$$\pi N(T)=\Delta_L arg\xi(s)$$

[9] (8-paragrafdagi (1)ga qarang) formulani aniqlash uchun

$$S(T)=O(log\,T) \qquad (2.14)$$

o‘rinli ekanligini ko‘rsatish yetarlidir. Bu bizning topilgan ζ −funksiya bilan bog‘liq bo‘lgan baholashdir. $arg\zeta(2)=0$ bo‘lganligi uchun, $S(T)$ ni quyidagicha kiritish mumkin

$$\pi S(T)=\Delta_L \arg\zeta\left(\frac{1}{2}+iT\right).$$

Bu ko‘rinish argument L bo‘ylab tekis yoki gorizontal chiziq bo‘ylab $+\infty+iT$ dan $\frac{1}{2}+iT$ gacha tekis harakatlanganda nolga teng bo‘lsa o‘rinli bo‘ladi. Buni inobatga olgan holda (2.13) dagi $\frac{7}{8}$ o‘zgarmasni tashlab yuborish tabiiy hol. (2.14) ni hosil qilish uchun quyidagi lemmani keltiramiz.

2.1-lemma. Agar $\rho=\beta+i\alpha$ $\zeta(s)$ ning notrivial nollardan bo‘lsa, bu holda katta T lar uchun quyidagi tenglik o‘rinlidir

$$\sum_\rho \frac{1}{1+(T-\gamma)^2}=O(\log T). \qquad (2.15)$$

Bu lemmani isbotlash uchun bizga [9] (13-paragraf, (4) ga qarang)dagi tengsizlikdan foydalanamiz. Tengsizlikka asosan

$$-Re\frac{\zeta'(s)}{\zeta(s)}<A\log t-\sum_p Re(\frac{1}{s-\rho}+\frac{1}{\rho})$$

bunda $1\leq\sigma\leq 2$ va $t\geq 2$. Faraz qilaylik bu formulada $s=2+iT$ bo‘lsin. Bu nuqtada $\left|\zeta'/\zeta\right|$ chegaralangan bo‘lganligi uchun

$$\sum_\rho Re(\frac{1}{s-\rho}+\frac{1}{\rho})<A\log T,$$

ko‘rinib turibdiki ikkita qatorning hadlari musbat va

$$Re\frac{1}{s-\rho}=\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2}\geq\frac{1}{4+(T-\gamma)^2}$$

bo‘lganligi uchun, lemma isbotlandi.

2.1-lemmadan quyidagi xulosalar kelib chiqadi:

a) $T-1<\gamma<T$ shartni qanoatlantiruvchi nollar soni $O(log\,T)$ ga teng.

b) γ berilgan intervaldan tashqarida yotgan holda $\sum(T-\gamma)^{-2}$ yig‘indining qiymati $O(log\,T)$ ga teng.

Yana shunday xulosa kelib chiqadiki (nol mos tushmagan) T lar $-1\leq\sigma\leq 2$ uchun

$$\frac{\zeta'(s)}{\zeta(s)}=\sum_{p}{}^{'}\frac{1}{s-\rho}+O(\log t) \tag{2.16}$$

bunda yig‘indi $|t-\gamma|<1$ shartni qanoatlantirishi kerak.

Haqiqatdan ham [9] (12-paragraf)dagi (8) dan bu tenglik ayrilsa quyidagi tenglik kelib chiqadi

$$\frac{\zeta'(s)}{\zeta(s)}=O(\log t)+\sum_{p}\left(\frac{1}{s-p}+\frac{1}{2+it-\rho}\right).$$

$|\gamma-t|\geq 1$ shartni qanoatlantiruvchi hadlari uchun $|2+it-\rho|\geq 1$ va b) xulosaga asosan ular yig‘indisi $O(log\,t)$ ga teng. a) xulosaga ko‘ra ularning soni $O(log\,t)$ ga teng.

S(T) uchun (2.14) baholashdan osongina (2.16) kelib chiqadi. Haqiqatdan ham *S(T)* ta’rifidan

$$\pi S(T)=O(1)-\int_{\frac{1}{2}+iT}^{2+iT} Im\left[\zeta'(s)/\zeta(s)\right]ds$$

O(1) had $\sigma=2$ to‘g‘ri chiziq bo‘ylab integrallash natijasida hosil bo‘ladi. Keyin

$$\int_{\frac{1}{2}+iT}^{2+iT} Im(s-p)^{-1}ds=\Delta\arg(s-\rho)$$

va absolyut qiymati π dan oshmaydi.

a) xulosaga ko‘ra (2.16) tenglikning yig‘indi qatnashgan hadlar soni $O(\log T)$

ga teng. Demak (2.14) o‘rinlidir.

Shunday qilib $N(T)$ uchun asimtotik formula isbotlandi. Bu formuladan, agar $\gamma \geq 0$ bo‘lganda $\gamma_1, \gamma_2, \ldots$ o‘sish tartibida nomerlangan bo‘lsa

$$\gamma_n \sim 2\pi n / \log n \text{ da } n \to \infty$$

$\gamma_{n+1} - \gamma_n \to 0$ kelib chiqmaydi. Bu natijani 1924-yil Littlvud tomonidan topilgan.

$N(T)$ uchun formula, agar H qandaydir musbat sondan katta bo‘lsa

$$N(T+H) - N(T) > A \log T \qquad (T > T_0)$$

ekanligini ko‘rsatadi. Titchmarsh bundan ko‘ra kuchli natijaga erishgan, bu natijaga ko‘ra yuqoridagi tengsizlik qandaydir A o‘zgarmas H ga bog‘liq bo‘lgan a) uchun o‘rinlidir.

$S(T)$ ga nisbatan Littlvud

$$\int_0^T S(t)\, dt = O(log\, T)$$

ekanligini isbotlagan.

Yuqorida keltirilgan natija, agar (2.13) dagi $\frac{7}{8}$ o‘zgarmas tashlab yuborilsa hosil bo‘lmasdi.

$\zeta(s)$ ning nollari haqidagi bu faktlar va boshqa natijalarning isboti (qarang: Titchmar, 9-bob).

II.3-§. $\psi(x)$ funksiya uchun aniqlashtirilgan formula

Biz bu paragrafda $\psi(x)$ uchun [9] (8-paragraf)da keltirilgan Mangolt formulasini isbotlaymiz. Eslatib o‘tamiz

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p$$

x tub sonning darajasi bo‘lganda bu funksiya uzilishga ega, shuning uchun formula o‘rinli bo‘lishi uchun bu nuqtalardagi chap va o‘ng qiymatlarining ta’rifini aniqroq kiritish zarur bo‘lib qoladi. Boshqacha aytganda, x tub sonning darajasi bo‘lmagan holda $\psi(x)$ ga teng bo‘lgan $\psi_0(x)$ funksiyani kiritamiz va aks holda

$$\psi(x) - \frac{1}{2}\, \Lambda(n)$$

bo‘lsin. Formula bo‘yicha $x > 1$ bo‘lganda

$$\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}), \qquad (2.17)$$

bunda $\zeta(s)$ funksiyaning notirivial nollarning ρ bo`yicha simmetrik yig‘indisi ko`rinishida olinganligi tushuniladi, ya’ni

$$\lim_{n\to\infty} \sum_{|\gamma|<T} \frac{x^{\rho}}{\rho}$$

[9] (12-paragraf, (8) va (10)ga qarang)ga asoslanib $\frac{\zeta'(0)}{\zeta(0)}$ o‘zgarmas $\log 2\pi$ ga teng ekanligini ko‘rsatishimiz mumkin[14]. Formuladagi oxirgi hadi $\zeta(s)$ funksiyaning $\omega = -2, -4, -6, \ldots$ nuqtalaridagi trivial nollariga - $\sum_{\omega} x^{\omega}/\omega$ ga ekvivalentdir.

Ba’zi bir kichik qiymatlarni bartaraf qilish uchun $x \geq 2$ deb faraz qilamiz, lekin yuqorida keltirilgan formula $x > 1$ da o‘rinlidir.

Bunga o‘xshash formulalarni isbotlash usuli Riman tomonidan yaratilgan edi. Isbotning asosiy g‘oyasi quyidagi ayirma

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} = \begin{cases} 0, & agar\ 0 < y < 1, \\ \frac{1}{2}, & agar\ \ y = 1 \\ 1, & agar\ \ y > 1 \end{cases} \qquad (2.18)$$

bunda $c > 0$, shu integral yordamida $n \geq x$ lar uchun $y = \frac{x}{n}$ deb qabul qilgan holda Dirixle qatorini tashlab ketildi.

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\zeta'(s)/\zeta(s),$$

natijada

$$\psi_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)}\right] \frac{x^s}{s} ds$$

agar vertikal integrallash yo‘lini chapga cheksizlikka o‘tkazilsa, bu holda $\psi_0(x)$ uchun $[-\zeta'(s)/\zeta(s)]\, x^s/s$ funksiyaning qutbdagi chegirmalar yig‘indisini hosil qilamiz. *S=1* nuqtada $\zeta(s)$ ning qutbi x ni beradi; $s = 0$ nuqtadagi $1/_S$ qutbi $-\zeta'(0)/\zeta(0)$ ni beradi, $\zeta(s)$ funksiyaning har qanday ρ nol esa , trivial yoki notrivial bo‘lishidan qat’iy nazar $-x^{\rho}/\rho$ ni beradi.

Isbotlashni amalga oshirish uchun, avval $c - i\infty$ dan $c + i\infty$ gacha bo‘lgan intervalni qarashimiz kerak va bu yo‘lni to‘g‘ri to‘rtburchakning chap tomoniga kattalashadigan tomon deb bilamiz. T ni extiyotkorlik bilan tanlab olish kerak, uni to‘rtburchakning gorizontal tomonlari $\zeta(s)$ning kritik polosasidagi nollarini aylanib o‘tishi zarur. Bunday fikrlash orqali (2.17)ning oxirgi ko‘rinishiga ega bo‘lamiz va bu ko‘rinishni qoldiq hadning baholashi (2.17) dan ancha yaqinroqdir. Quyidagi lemmani isbotlaymiz.

2.2-lemma. O‘ng tomoni (2.18) bilan aniqlangan y bo‘yicha $\delta(y)$ funksiya berilgan bo‘lsin va

$$I(y,T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds$$

bu holda $y > 0$, $c > 0$ va $T > 0$ bo‘lganida

$$|I(y,T) - \delta(y)| < \begin{cases} y^c \min(1, T^{-1}|\log y|^{-1}), & agar\ \ y \neq 1, \\ cT^{-1}, & agar\ y = 1. \end{cases}$$

Isboti. Faraz qilaylik $0 < y < 1$, $\sigma \to +\infty$ ga y^s/s funksiya t bo‘yicha 0 ga tekis yaqinlashsin. Demak vertikal to‘g‘ri chiziq bo‘yicha integrallash gorizontal to‘g‘ri chiziq bo‘yicha ikkita integralga ajratishimiz mumkin

$$I(y,T) = -\frac{1}{2\pi i}\int_{c+iT}^{\infty+iT} \frac{y^s}{s} ds + \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{c-iT}^{\infty-iT} \frac{y^s}{s} ds \tag{2.19}$$

keyin

$$\left|\int_{c+iT}^{\infty+iT} \frac{y^s}{s} \mathrm{ds}\right| \leq \frac{1}{T}\int_{c}^{\infty} y^{\sigma} d\sigma = \frac{y^c}{T|\log y|} \tag{2.20}$$

Ikkinchi integral uchun ham shunga o‘xshash formula o‘rinlidir. Ikkinchi tengsizlikni osonlikcha isbotlash mumkin, agarda vertikal yo‘lni, to‘g‘ri chiziqning o‘ng tomonda joylashgan markazi nol nuqtada doiraga almashtirilsa. Doira radiusi $R = \sqrt{c^2 + T^2}$ va uning yoyi uchun $|y^s| \leq y^c$ va $|s| = R$, demak

$$|I(y,T)| \leq \frac{1}{2\pi}\pi R\frac{y^c}{R} < y^c$$

$y > 1$ dagi holda isbot shunga o‘xshash bajariladi. Faqat to‘g‘ri to‘rtburchak va doira yoyi to‘g‘ri chiziqning chap tomoniga olinadi. Bu holdagi kontur $s = 0$ nuqtadagi $1 = \delta(y)$ chegirmali qutbni o‘z ichiga qamrab oladi, $y = 1$ holat qoladi. Bu holda aniq hisob kitob bajariladi. $s = c + iT$ bo‘lganda

$$I(1,T) = \frac{1}{2\pi}\int_0^T \frac{2c}{c^2+t^2}dt = \frac{1}{\pi}\int_0^{T/c}\frac{du}{1+u^2} = \frac{1}{2} - \frac{1}{\pi}\int_{T/c}^{\infty}\frac{du}{1+u^2} \qquad (2.21)$$

oxirgi integral ${}^{c}/_{T}$ dan kichikdir. Shunday qilib lemma isbotlandi.

2.2-lemmani $\psi_0(x)$ ga qo‘llasak quyidagi natijaga erishamiz

$$|\psi_0(x) - J(x,T)| < \sum_{\substack{n=1\\ n\neq x}} \Lambda(n)(x/n)^c \min(1, T^{-1}|\log x/n|^{-1}) + cT^{-1}\Lambda(n), \qquad (2.22)$$

bunda $c > 1$ va

$$J(x,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]\frac{x^s}{s}ds \qquad (2.23)$$

eslatib o‘tish kerakki, faqat x tub sonning darajasi bo‘lganda $\Lambda(n)$ga bog‘liq hadi mavjud bo‘ladi. $c = 1 + (\log x)^{-1}$ deb olamiz, chunki bu holda yaxshi, aniqroq baholashga ega bo‘lamiz[15]. Eslatish kerakki $x^c = ex$ (2.22) ning o‘ng tomonidagi qatorni baholashimiz kerak va boshqa $n \leq \frac{3}{4}x$ yoki $n \geq \frac{5}{4}x$ shartlarni qanoatlantiruvchi hadlarni qaraymiz. Bu hadlar uchun $|\log {}^{x}/_{n}|$ o‘zining musbat quyi chegarasiga ega

$$\ll xT^{-1}\sum_{n=1}^{\infty}\Lambda(n)n^{-c} = xT^{-1}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] \ll xT^{-1}(\log x) \qquad (2.24)$$

Keyin $\frac{3}{4}x < n < x$ shartlarni qanoatlantiruvchi hadlarni tekshiramiz, faraz qilaylik x-tub sonning eng katta darajasi bo‘lsin va u x dan kichik yoki $\frac{3}{4}x < x_1 < x$, chunki aks holda qaralgan hadlar mavjud emas $n = x_1$ bo‘lganda

$$\log\frac{x}{n} = -\log(1 - \frac{x-x_1}{x}) \geq \frac{x-x_1}{x}. \qquad (2.25)$$

Demak bu qatorningn hadlari uchun quyidagi baholash o‘rinlidir

$$\ll \Lambda(x_1) min\left[1, \frac{x}{T(x-x_1)}\right] \ll (\log x) min\left[1, \frac{x}{T(x-x_1)}\right]. \tag{2.26}$$

1) Bu yerda keyinchalik $A \ll B$, $A = O(B)$ ga teng kuchli bo‘lgan Vinogradov simvolini ishlatamiz, qolgan hadlar uchun $n = x_1 - \nu$ deb olinsa, bunda

$$0 < \nu < \frac{1}{4}x$$

$$\log\frac{x}{n} \geq \log\frac{x_1}{n} = -\log\left(1 - \frac{\nu}{x_1}\right) \geq \frac{\nu}{x_1} \tag{2.27}$$

demak bu hadlar yig‘indisiga, quyidagi miqdorni qo‘shadi

$$\ll \sum_{0<\nu<\frac{1}{4}x} \Lambda(x_1 - \nu) T^{-1} x_1/\nu \ll x T^{-1} (\log x)^2. \tag{2.28}$$

$x < n < \frac{5}{4}x$ shartni qanoatlantiruvchi hadlar ham quyiga o‘xshash baholanadi, faqat x_1 ni x_2 ga almashtirganda eng kichik darajasi x dan katta bo‘lishi kerak. x dan tub sonning darajasigacha bo‘lgan masofani $< x >$ orqali belgilash qulaydir. Barcha baholashlarni yig‘ib olganda (2.22) dan

$$|\psi_0(x) - J(x,T)| \ll \frac{x(\log x)^2}{T} + \log x \min(1, \frac{x}{T<x>}). \tag{2.29}$$

kelib chiqadi.

Keyingi qadamda (2.23) da integrallashning vertikal yo‘lini uchlari

$$c - iT, c + iT, U + iT, \quad -U \; - iT,$$

nuqtalarda joylashgan to‘g‘ri to‘rtburchakka almashtiramiz, bunda U-katta toq sondir. Shunday qilib, chap vertikal tomon $\zeta(x)$ ning trivial nollar o‘rtasidan o‘tadi. Integral ostidagi funksiyaning qutbdagi chegirmalar yig‘indisi (to‘rtburchak ichidagi)

$$x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{0<2m<U} \frac{x^{-2m}}{2m} \quad \text{teng.} \tag{2.30}$$

T ni tanlab olganda qo‘shimcha kiritish kerak. [9] (15-paragraf)dagi lemmaga ko‘ra T uchun $|\gamma - T| < 1$ shartni qanoatlantiruvchi nollar soni $\ll \log T$ ga teng. Bu nollarning ordinatalar orasidagi masofa $\gg (\log T)^{-1}$ ga teng bo‘lishi kerak, demak shunday T tanlash mumkinki, barcha $\beta + i\gamma$ nollar uchun $|\gamma - T| \gg (\log T)^{-1}$ bo‘ladi.

[9] (15-paragraf)dagi natijaga ko‘ra $s = \sigma + iT$ va $-1 \leq \sigma \leq 2$ uchun

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\gamma - T|<1} \frac{1}{s-p} - O(\log T). \tag{2.31}$$

T ni tanlashga ko‘ra har bir had $\ll \log T$ bo‘ladi va ular soni $\ll \log T$, shuning uchun mos bo‘lgan gorizontal to‘g‘ri chiziqlar uchun quyidagi baholash o‘rinlidir

$$\frac{\zeta'(s)}{\zeta(s)} = O(log^2 T), \text{ bunda } -1 \le \sigma \le 2.$$

Shuning uchun bunday σ ga gorizontal bo‘yicha integral

$$\ll log^2 T \int_{-1}^{c} \left|\frac{x^s}{s}\right| d\sigma \ll \frac{log^2 T}{T} \int_{-\infty}^{c} x^\sigma d\sigma \ll \frac{x log^2 T}{T \log T} \tag{2.32}$$

endi $-u \le \sigma \le 1$ gorizontal kesma va $\sigma = -u$vertikal to‘g‘ri chiziq chiziq bo‘yicha integralni baholash qolyabdi. Bular uchun $\sigma \le -1$dagi $|\zeta'/\zeta|$ baholash kerak. Shuning uchun markazlari $s = -2, -4, -6, ...$ trivial nollari joylashgan radiusi $\frac{1}{2}$ ga teng doira uchun quyidagi baholash o‘rinli ekanligini ko‘rsatamiz

$$|\zeta'(s)/\zeta(s)| \ll \log(2|s|). \tag{2.33}$$

Bundan, qolgan gorizontal to‘g‘ri chiziq

$$\ll \frac{\log 2T}{T} \int_{-U}^{-1} x^\sigma d\sigma \ll \frac{T \log U}{U x^U},$$

miqdor (2.26) ga ko‘ra kichikdir. Vertikal to‘g‘ri chiziq bo‘yicha integral esa

$$\ll \frac{\log 2U}{U} \int_{-T}^{T} x^{-U} dt \ll \frac{T \log U}{U x^U}$$

ga teng bo‘lib, $U \to \infty$ ga 0 ga intiladi. (2.26), (2.24) va (2.25) baholashlarni yig‘ib olsak ($U \to \infty$) quyidagi natijaga erishamiz

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) + R(x,T) \tag{2.34}$$

Bunda

$$R(x,T) \ll \frac{x log^2(xT)}{T} + (\log x)\min(1, \frac{x}{T<x>}) \tag{2.35}$$

(2.33) baholash quyidagi funksional tengsizlikdan kelib chiqadi, [9] 10-paragrafdagi (4)ning nosimmetrik ko‘rinishini yozamiz

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\left(\cos\frac{1}{2}\pi s\right)\Gamma(s)\zeta(s) \tag{2.36}$$

agar $1 - \sigma \le -1$ bo‘lganda o‘ng tomondan funksiya faqat $\sigma \ge 2$ bo‘lganda qaraladi.

O‘ng tomondan funksiyaning logarifmik hosilasi o‘zgarmas qo‘shiluvchi aniqligida

$$-\frac{1}{2}\pi\, tg\, \pi s + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}$$

ga teng[15].

Birinchi had chegaralangan, agar

$$|1-(2m+1)| \geq \frac{1}{2}$$

yoki

$$|(1-s)+2m| \geq \frac{1}{2}$$

bo`lsa.

Ikkinchi had $\ll \log|s|$ va bu degani $\sigma \geq 2$ da $\ll \log 2|1-s|$. Uchinchi qo‘shiluvchi ham chegaralangan, demak (2.33) isbotlandi.

(2.34) va (2.35) natijalar (2.17) formulaning aniq ko‘rinishini ifodalaydi. Ixtiyoriy $x \geq 2$ uchun $T \to \infty$ da $R(x,T) \to 0$ va bu tufayli (2.17) tenglik bajariladi. Tub sonning darajasini o‘z ichiga olmasligi ixtiyoriy yopiq kesmada yaqinlashishi tekis hisoblanadi, lekin qolgan qismda tekis emas, chunki $\psi_0(x)$ x tub sonning darajasi bo‘lsa, bu nuqtada uzilishga ega.

(2.34) va (2.35) lar isbotlanganda T ga shart qo‘yilgan edi, lekin uni inobatga olmaslik mumkin. T ning chegaralangan miqdorga o‘zgarishi ρ bo‘yicha $O(\log T)$ hadlarning yig‘indisining o‘zgarishiga teng kuchlidir. Shu hadlarning $O(X/T)$ ga teng. Demak yig‘indi $O(x(\log T)/T)$ gach o‘zgaradi, bu esa (2.35) ning o‘ng tomoniga ta’siri yo‘q. Keyingi uchun, aytib ketish kerakki x-butun $< x > \geq 1$ bo‘lganida (2.35) sodda ko‘rinishga ega

$$|R(x,T)| \ll x(\log xT)^2 T^{-1}$$

(2.34) va (2.35) munosabatlar $1 < x < 2$ bo‘lganida ham o‘rinli bo‘ladi, faqat bu holda $R(x,T)$ uchun baholash murakkab ko‘rinishga ega.

Ikkinchi bob yuzasidan xulosa

Ushbu dissertatsiyaning ikkinchi bobi $\zeta(s)$-funksiyaning logarifmik hosilasini nollari bo‘yicha qatorga yoyish. $\zeta(s)$-funksiyaning nollari haqida teoremalar keltirilgan. $N(T)$funksiya uchun formulaning qoldiq hadiga kiruvchi o‘zgarmasning qiymati aniqlangan. $\psi(x)$ funksiya uchun baholar olingan.

III-BOB. $\psi(x,\chi)$ FUNKSIYASI UCHUN ANIQ FORMULA

III.1-§. *L*-funksiyaning logarifmik hosilasini nollari bo‘yicha qatorga yoyish.

Yuqorida 1.3-paragrafda isbotlangan teoremaning natijasidan ko‘rinadiki, $\chi(mod k)$primitiv xarakter bo‘lsa, $L(s,\chi)$ funksiya $Res < 0$ yarim tekislikda faqat haqiqiy nollarga ega, bu nollar $\Gamma\left(\frac{s}{2}\right)$ va $\Gamma\left(\frac{s+1}{2}\right)$larning qutblaridan iboratdir. $L(s,\chi)$ning bu nollariga uning trivial nollari deyiladi. Shuningdek $s=0$ nuqtadagi noli ham trivial nollarga kiradi. Bu trivial nollardan tashqari $L(s,\chi)$ funksiya kritik yo‘lak $0 \le \operatorname{Re} s \le 1$ da cheksiz ko‘p trivial bo‘lmagan nollarga ham ega.

Agar $\chi(mod k)$primitiv xarakter va

$$\delta = \begin{cases} 0, & agar\, \chi(-1) = +1 \text{ bo‘lsa;} \\ 1, & agar\, \chi(-1) = -1 \text{bo‘lsa ;} \end{cases}$$

$$\xi(s,\chi) = \left(\frac{\pi}{k}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi)$$

bo‘lsa (1.3-§ ga qarang), ushbu teorema o‘rinli.

3.1-teorema. Agar χ – primitiv xarakter bo‘lsa, $\xi(s,\chi)$funksiya birinchi tartibli butun funksiya bo‘lib, $0 \le \operatorname{Re}\rho_n \le 1$, $\rho_n \ne 0$ shartni qanoatlantiruvchi cheksiz ko‘p nollarga ega hamda

$$\sum_{n=1}^{\infty} |\rho_n|^{-1}$$

qator uzoqlashuvchi va

$$\sum_{n=1}^{\infty} \frac{1}{|\rho_n|^{1+\varepsilon}}$$

qator esa ixtiyoriy $\varepsilon > 0$ uchun yaqinlashuvchidir. $\xi(s,\chi)$ funksiyaning nollari $L(s,\chi)$ funksiyaning trivial bo‘lmagan nollaridir[16].

Bu teoremani isbotlashda biz quyidagi analitik funksiyalar nazariyasiga doir lemmadan, ya’ni chekli tartibli butun funksiyani cheksiz ko‘paytma ko‘rinishda ifodalashga doir ushbu tasdiqdan foydalanamiz.

3.1-lemma. Agar $G(s)$ chekli α tartibli butun funksiya va$G(0) \ne 0$bo‘lsin, s_n

esa $G(s)$ ning barcha nollari ketma-ketligi bo‘lib

$$0<|s_1|\le|s_2|\le\cdots\le|s_n|\le\cdots$$

shartni qanoatlantirsa, u holda s_n ketma-ketlik chekli yaqinlashish ko‘rsatkichi $\beta\le\alpha$ ga ega bo‘ladi va

$$G(s)=e^{g(s)}\prod_{n=1}^{\infty}\left(1-\frac{s}{s_n}\right)e^{\frac{s}{s_n}+\frac{1}{2}\left(\frac{s}{s_n}\right)^2+\ldots+\frac{1}{p}\left(\frac{s}{s_n}\right)^p},$$

bu yerda $p\ge0$

$$\sum_{n=1}^{\infty}\frac{1}{|s_n|^{p+1}}<\infty$$

tengsizlikni qanoatlantiruvchi eng kichik butun son, $g(s)$ esa $g-$darajali$(g\le\alpha)$ko‘r had, $\alpha=\max(g,\beta)$. Agarda bundan tashqari ixtiyoriy $c>0$ uchun shunday bir $|s|=r_n$, $n=1,2,\ldots,\ (r_n\to+\infty)$ ketma-ketlik mavjud bo‘lib

$$\max|G(s)|>e^{cr_n^{\alpha}}$$

tengsizlik bajarilsa, u holda $\alpha=\beta$

$$\sum_{n=1}^{\infty}\frac{1}{|s_n|^{\beta}}$$

qator uzoqlashadi[16].

Isboti. $\operatorname{Re}s\ge\frac{1}{2}$ bo‘lganda 1.3.4-lemmaning natijasiga ko‘ra

$$|L(s,\chi)|\le2|s|\varphi(k)<2|s|k\ ,$$

bunda $\xi(s,\chi)$ funksiyaning aniqlanishiga ko‘ra

$$|\xi(s,\chi)|\le2k^{\frac{\sigma}{2}+\frac{3}{2}}|s|\left|\Gamma\left(\frac{s+\delta}{2}\right)\right|<<k^{\frac{\delta}{2}+\frac{3}{2}}e^{c|s||\ln|s||}$$

Bu yerda

$$\Gamma(s)\ll e^{c|s||\ln|s||}$$

ekanligidan foydalandik.

$\xi(s,\chi)$ ning bu oxirgi bahosi (1.3.7)-funksional tenglama va

$$\left|\frac{i^{\delta}\sqrt{k}}{\tau(\chi)}\right|=1$$

ga asosan $\mathrm{Re}\, s<\frac{1}{2}$ bo‘lganda ham o‘rinli. Bundan tashqari $\xi(0,\chi)\neq 0.$ $s\to\infty$ da $\ln\Gamma(s)\sim s\ln s$ bo‘lganligi sababli yuqoridagi 2.1-lemmadan teoremaning birinchi tasdig‘i kelib chiqadi. $\mathrm{Re}\, s>1$ da $L(s,\chi)\neq 0$ bo‘lgani uchun (1.3.7) dan $\mathrm{Re}\, s<0$ da $\xi(s,\chi)\neq 0$ ekanligi,ya’ni $\xi(s,\chi)$ funksiyaning nollari $L(s,\chi)$ funksiyaning $0\le \mathrm{Re}\, s\le 1$ yo‘lakdagi nollari bo‘ladi. Teorema to‘la isbot bo‘ldi.

3.1-natija. Ushbu formula o‘rinli

$$\xi(s,\chi)=e^{A+Bs}\prod_{n=1}^{\infty}\left(1-\frac{s}{\rho_n}\right)e^{\frac{s}{\rho_n}}, \tag{3.1}$$

bu yerda $A=A(\chi),\ \ B=B(\chi)-$ o‘zgarmas sonlar.

(1.3.7) funksional tenglamadan $L(s,\chi)$ funksiyaning trivial bo‘lmagan nollari $\mathrm{Re}\, s=\frac{1}{2}$ to‘g‘ri chiziqqa nisbatan simmetrik joylashgan. Bundan keyin biz nollar $\rho_n, n=1,2,\ldots$ ni ularning mavhum qismlari absolyut qiymatlarining o‘sib borishi tartibida nomerlangan deb qaraymiz.

Endi $B=B(\chi)$ ning $L(s,\chi)$ ning trivial bo‘lmagan nollari orasidagi bog‘lanishni keltirib chiqaramiz. Buning uchun avvalo (3.1) ning ikkala tomonini logarifmlab, keyin differensiallaymiz, u holda quyidagiga ega bo‘lamiz:

$$\ln\xi(s,\chi)=(A+Bs)+\sum_{n=1}^{\infty}\ln\left(1-\frac{s}{\rho_n}\right)+\frac{s}{\rho_n}$$

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)}=B(\chi)+\sum_{n=1}^{\infty}\frac{-\frac{1}{\rho_n}}{1-\frac{s}{\rho_n}}+\frac{1}{\rho_n}=B(\chi)+\sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right).$$

Bundan va (1.3.7) dan

$$\frac{\xi'(0,\chi)}{\xi(0,\chi)} = B(\chi) = \frac{-\xi'(1,\overline{\chi})}{\xi(1,\overline{\chi})} = -\sum_{n=1}^{\infty}\left(\frac{1}{1-\overline{\rho}_n} + \frac{1}{\rho_n}\right) - \mathrm{B}(\overline{\chi})$$

ni hosil qilamiz. Bunda $L(\rho_n,\chi) = L(1-\rho_n,\overline{\chi}) = L(\overline{\rho}_n,\overline{\chi}) = L(1-\overline{\rho}_n,\chi) = 0$, shuning uchun ham ρ_n va $1-\overline{\rho}_n$ lar $L(s,\chi)$ning nollari va

$$\operatorname{Re} B(\chi) + \frac{1}{2}\sum_{n=1}\left(\frac{1}{\rho_n} + \frac{1}{\overline{\rho}_n}\right) = 0. \tag{3.2}$$

tenglikka ega bo‘lamiz.

Ma’lumki, Riman ζ(s) funksiyaning nollari to‘g‘risidagi quyidagi gipotezani ilgari surgan edi. ζ(s) funksiyaning barcha trivial bo‘lmagan nollari $\sigma = \frac{1}{2}$ to‘g‘ri chiziq ustida yotadi. Bu tasdiqqa Riman gipotezasi deyiladi. Keyinchalik bu tasdiq $L(s,\chi)$ning nollari uchun umumlashtirilib ”$L(s,\chi)$funksiyaning barcha trivial bo‘lmagan nollari $\sigma = \frac{1}{2}$ to‘g‘ri chiziq ustida yotadi” – degan tasdiq vujudga keldi va uni Rimanning $L(s,\chi)$ nollari haqidagi umumlashgan gipotezasi deb atala boshlandi.

Hozirgacha bu ikkala tasdiq ham to‘la isbotlanmagan. Lekin olingan keyingi natijalarning barchasi [17] shu Rimanning gipotezalari deb ataluvchi tasdiqlarning to‘g‘riligini isbotlashga asos bo‘ladi.

Umuman $L(s,\chi)$ funksiyaning nollari to‘g‘risida quyidagi tasdiq Peydj tomonidan isbotlangan: agar $\chi(\operatorname{mod} q)$ bo‘lib $q \le T$ bo‘lsa, u holda barcha $L(s,\chi)$ funksiyalar

$$\sigma > 1 - \frac{C_1}{\ln T}, \quad |t| \le T \tag{3.3}$$

Sohada kompleks nolga ega emas. (3.3) sohada $L(s,\chi)$ning faqat $\chi(\operatorname{mod} r)$ Primitiv xarakter uchun bitta haqiqiy nolga ega bo‘lishi mumkin, u nol $1-\beta$ quyidagi tengsizlik

$$\frac{\tilde{N}_1}{\ln T} < 1-\beta < \frac{C_2}{r^{\frac{1}{2}}\ln^2 r}. \tag{3.4}$$

ni qanoatlantiradi[18,19].

III.2-§. $N(T,\chi)$funksiya uchun formulaning qoldiq hadiga kiruvchi o‘zgarmasning qiymatini aniqlash

$\chi -q$ modul bo‘yicha Dirixle xarakteri bo‘lsin. Ma’lumki, $L(s\ \chi)$ bilan bog‘liq turli xil muammolar uchun $N(T,\chi)$ da ko‘pincha aniq formulalar [9] dagi 16 va 19-larda) qo‘llaniladi. Shuning uchun ko‘pgina sonlar nazariyasi konstantalarning son qiymatini aniqlashda aytib o‘tilgan aniq formulalarga kiritilgan konstantalarning son qiymatlarini bilish muhimdir[20].

Ayniqsa, ular Montgomer va Vogan, δ va $X_o(\delta)$ konstantalar qiymatlarini aniqlashda kerak.

Faraz etaylik, $N(T,\chi)$ Dirixle L-funksiyaning $\frac{1}{2}\leq\sigma\leq 1, |t|\leq T$ oraliqdagi nollari soni bo‘lsin.

3.2-teorema. $T\geq T_o$uchun quyidagi formula o‘rinli:

a) agar χ -q≥3 modul bo‘yicha primitiv xarakterga ega bo‘lsa, u holda

$$N(T,\chi)=\frac{\mathrm{T}}{\pi}\log\frac{qT}{2\pi}-\frac{T}{\pi}+28{,}6836\theta_1\log qT$$

agar $\chi=\chi_o^*-1$ modul bo‘yicha bosh xarakterga ega bo‘lsa, u holda

$$N(\chi_\circ^*,T)=\frac{\mathrm{T}}{\pi}\log\frac{T}{2\pi}-\frac{T}{\pi}+29{,}8373\theta_2\log T$$

b) agar χ -$q\geq 3$ modul bo‘yicha ixtiyoriy xarakterga ega bo‘lsa, u holda

$$N(T,\chi)=\frac{\mathrm{T}}{\pi}\log\frac{qT}{2\pi}-\frac{T}{\pi}+7{,}1674\theta_3T\log q$$

bu yerda $|\theta_i|\leq 1$, *i=1,2,3.*

Ushbu teoremani ibotlash uchun quyidagi lemmalar kerak.

3.2-lemma. $-\pi<args<\pi$ Uchun quyidagi formula o‘rinli:

$$\log\Gamma(s)=(s-\frac{1}{2})\log s-s+\frac{1}{2}\log 2\pi+r(\frac{1}{s})$$

bu yerda

$$r(\frac{1}{s})|\leq\frac{1}{12|s|}(1+\frac{1}{30|s|^2}+\frac{1}{105|s|^4}$$

Isboti. $|args|<\pi$ uchun quyidagi tenglik o‘rinli ekani ma’lum, [9] dagi

(6.1.42) ga qarang) :

$$r\left(\frac{1}{s}\right) = log\,\Gamma(s) - \; log(s-\tfrac{1}{2})\,log\,s - s + \tfrac{1}{2}log\,2\pi = R\left(\tfrac{1}{2}\right) + \sum_{m=1}^{n} \frac{B_{2m}}{2m(2m-1)s^{2m-1}},$$

bu yerda B_{2m}-Bernullining *2m*-soni va

$$|R\left(\frac{1}{2}\right)| \leq \frac{|B_{2n+2}|K(s)}{(2n+1)(2n+2)|s|^{2n+1}}$$

$K(s) - u \geq 0$ uchun $|s^2(u^2+s^2)|$ yuqori chegara. $K(s) \leq 1$ bo‘lim, $n = 2$ deb olamiz va lemma tasdig‘iga ega bo‘lamiz.

3.2- natija. $\mathscr{b}$ doimiysi (($|\mathscr{b}| < |s|$) va $|args| < \pi$ uchun quyidagi tenglik o‘rinli:

$$\log\Gamma(s + \mathscr{b}\,) = (s + \mathscr{b}\, - \frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + r_1(\frac{1}{s})$$

bu yerda

$$\left|r_1\left(\frac{1}{s}\right)\right| \leq \frac{|3\mathscr{b}^2 + \mathscr{b}|}{2(|s| + |\mathscr{b}|)} + |r(\frac{1}{s+\mathscr{b}})|$$

3.3-natija. $|args| < \pi - \delta(\delta > 0)$ uchun quyidagi o‘rinli:

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + r_\delta(\frac{1}{s})$$

bu yerda

$$|r_\delta(\frac{1}{s})| \leq \frac{\delta}{2|s|\sin\delta} < \frac{\pi}{4|s|}$$

Isboti. $log\,\Gamma(s)$ uchun quyidagi munosabat o‘rinli ([10] ning 28-betiga qarang):

$$\log\Gamma(s) = (s - \tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi + \int_0^\infty \frac{[u]-u+\frac{1}{2}}{u+s}du \qquad (3.5)$$

Bu yerdan

$$|\frac{\Gamma'}{\Gamma}(s) - \log s + \frac{1}{2s}| = |\int_0^\infty \frac{[u]-u+\frac{1}{2}}{(u+s)^2}du| \leq \frac{1}{2}\int_0^\infty \frac{[u]-u+\frac{1}{2}}{u^2+|s|^2-2u\cos\delta} = \frac{\delta}{2|\delta|\sin\delta} < \frac{\pi}{4|s|}$$

(3.5) munosabatdagi integral belgisi ostida differensiatsiyaning qonuniyligini tekshirish qiyin emas.

3.2-teoremaning isboti. $\chi - q$ modul bo‘yicha primitiv xarakterga ega bo‘lsin va

$$\xi(s,\chi) = (\frac{q}{\pi})^{\frac{1}{2}(s+a)} + \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) \tag{3.6}$$

s uchun $\frac{5}{2} + iT, -\frac{3}{2} + iT$ nuqtalarda bo‘lgan R to‘g‘ri to‘rtburchak bo‘ylab harakatlanganda arg $\xi(s,\chi)$ ortishini ko‘rib chiqaylik. Bu to‘rtburchak $s = 0$ yoki $s = 1$ nuqtada faqat bitta trivial nol $L(s,\chi)$ni o‘z ichiga oladi va shuning uchun,

$$2\pi(N(T,\chi) + 1 = \Delta_R \arg \xi(s,\chi)$$

Quyidagi funksional tenglamaga ko‘ra,

$$\xi(1-s,\bar{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)}\ \xi(s,\chi)$$

Biz arg $\xi(\sigma + iT,\chi) = \overline{arg(\sigma + \imath T,\chi) + c}$ …. ga ega bo‘lamiz, bu yerda … ga bog‘liq emas, shuning uchun konturning chap yarmi bo‘ylab harakatlanayotganda argumentning o‘sishi o‘ng yarmi bo‘ylab harakatlanayotganda o‘sishga teng bo‘ladi[21]. Shunday qilib,

$$2\pi(N(T,\chi) + 1 = \Delta_R \arg \xi(s,\chi) = 2(\Delta_{R_1} \arg \xi(s,\chi)) \tag{3.7}$$

bu yerda $R_1 - R$ ning o‘ng yarmida o‘tayotgandagi argumentning o‘sishi.

1-natijadan foydalanib, $\Delta_{R_1} \arg \xi(s,\chi)$ ni hisoblaymiz:

$$\Delta_{R_1} \arg(\frac{q}{\pi})^{\frac{1}{2}(s+a)} =, \Delta_{R_1}(\frac{t}{2}\log\frac{q}{\pi}) = \mathrm{T}\log\frac{q}{\pi}$$

$$\Delta_{R_1} arg\Gamma\left(\frac{s+a}{2}\right) = \Delta_{R_1} Im \log\Gamma\left(\frac{s+a}{2}\right)$$

$$2\left\{Im \log\Gamma\left(\frac{1}{2}\left(\frac{1}{2} + a + iT\right)\right) - Im \log\Gamma(\frac{1}{2}(\frac{5}{2} + a))\right\}$$

$$= 2Im\left\{(-\frac{1}{4} + \frac{a}{2} + \frac{1}{2}iT)\log(\frac{1}{2}iT) - \frac{1}{2}iT + \frac{1}{2}\log 2\pi + r_1(\frac{1}{s})\right\}$$

$$= \log\frac{T}{2} - T + (2a-1)\frac{\pi}{4} + 2\theta_6|r_1(\frac{1}{s})|,\ |\theta_6| \leq 1$$

1-natijadan $T \geq T_0$ uchun quyidagi kelib chiqadi:

$$2|r_1(\frac{1}{s}| < 2\left\{\frac{15}{16(1-1{,}5T_0^{-1})} + \frac{1}{6}\left(1 + \frac{2}{15T_0^2} + \frac{16}{105T_0^4}\right)\right\}\frac{1}{T} = C_{24}\frac{1}{T}$$

Endi L(s,χ) argumentni ko‘rsatamiz.

$$\pi S(\chi,T) = \Delta_{R_1} arg\ \mathrm{L}(s,\chi)$$

ni belgilaymiz. Bu tenglikning o‘ng tomonini quyidagicha yozish mumkin:

$$\Delta_{R_1} arg\ \mathrm{L}(\mathrm{s},\chi) = 2\int_{5/2}^{5/2+it} Im\left\{\frac{\acute{L}}{L}(s,\chi)\right\}\mathrm{ds} - 2\int_{1/2}^{5/2+it} Im\left\{\frac{\acute{L}}{L}(s,\chi)\right\}\mathrm{ds}$$

Birinchi integral quyidagi bahoga ega

$$\left|2\int_0^T Im\sum_{n=1}^{\infty}\frac{\chi(n)\lambda(n)}{n^{5/2+it}}dt\right| \le 4\zeta\left(5/2\right) = 5{,}364$$

Endi ikkinchi integralni baholaylik. Shunday qilib,

$$\left|\int_{5/2}^{5/2+iT} Im(s-a)^{-1}ds\right| = |\Delta\arg(s-a)| \le \pi$$

bo‘lsa, u holda q≥ q_0 ,T≥ T_0lar uchun

$$\pi\mathcal{S}(\chi,T) \le C_{25}\mathrm{l}$$

ga ega bo‘lamiz, bu yerda C_{25} =4C_{22} +2$\pi C_{23} + 5{,}364 l_0^{-1}$.Demak, (3.7) dan quyidagi tenglik kelib chiqadi

$$N(T,\chi) = \frac{\mathrm{T}}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi} + \theta_1 C_{26}\mathrm{l}$$

bu yerda

$$C_{26} = \frac{1}{\pi}\left(C_{25} + (T_0 l_0)^{-1}C_{24}\right)$$

$$= 3{,}0385\left\{1 + \begin{pmatrix}(\gamma_0 + (2T_0)^{-1} + (12T_0^2)^{-1} +) \\ \frac{1}{2}\sqrt{1 + 12{,}25T_0^{-2}}\end{pmatrix}\mathcal{L}_0^{-1}\right\}$$

$$+(4{,}4969 + 0{,}5968(T_0 - 1{,}5)^{-1}) + 0{,}1061T_0^{-1} + 0{,}0142T_0^{-3} + 0{,}01617T_0^{-5})l_0^{-1}$$

(3.8)

Keling $\chi = \chi_0^*$ -1 modul bo‘yicha bosh xarakterga ega bo‘lsin. U holda L(s,χ_0^*)=$\zeta(s)$ va $\xi(s,\chi)$ ning o‘rniga

$$\xi(s,\chi) = (s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\zeta(s)$$

funksiyani ko‘rsatamiz.

Bu holda oldingi mulohazalar ([9]dagi 15-$ga\ qarang$) quyidagi formulaga olib keladi:

$$N(T,\chi_0^*) = \frac{\mathrm{T}}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + \theta_2 C_{27}\mathcal{L} \quad (3.9)$$

bu yerda

$$C_{27}=5{,}4843\left\{1+\left(\frac{(\gamma_0+(2T_0)^{-1}+(12T_0^2)^{-1}+)}{\frac{1}{2}\sqrt{1+4T_0^{-2}}}\right)\mathcal{L}_0^{-1}\right\}$$

$$+(21{,}7472+0{,}3786T_0^{-1}+0{,}7162T_0^{-2})+0{,}0091T_0^{-3}+0{,}0103T_0^{-5})\mathcal{L}_0^{-1}$$

Nihoyat, χ-primitiv xarakterga ega bo‘lmay, q_1 modul bo‘yicha $\chi_1$primitiv xarakterga ega bo‘lsin. L(s_1,χ_1) funksiya, L(s,χ) funksiyaning nollaridan tashqari, $p/_{q},\ p*q$ shu kabi ba’zi p lar bo‘yicha $1-\chi_1(p)p^{-s}$ uchun faqat s nuqtada nollarga ega, ya’ni

$$s=\frac{\log\chi_1(p)}{\log p}=i\frac{arg\chi_1(p)+2\pi n}{\log p}$$ (n- butun son)

$|t|\leq T$ sohadagi barcha bunday nuqtalar soni quyidagiga teng:

$$\frac{1}{2\pi}\sum_{p/_{q},\,p*q}(T\log p+1)\leq\frac{1}{2\pi}\left(1+\frac{1}{T_0\log 2}\right)T\log q$$

Natijada, bu holatda

$$N(T,\chi)=\frac{\mathrm{T}}{\pi}\log\frac{qT}{2\pi}-\frac{T}{\pi}+\theta_7\frac{1}{2\pi}\left(1+\frac{1}{T_0\log 2}\right)T\log q+C_{26}\theta_1 \mathrm{l}$$

yoki agar T$\geq T_0(\geq e)$, u holda

$$N(T,\chi)=\frac{\mathrm{T}}{\pi}\log\frac{qT}{2\pi}-\frac{T}{\pi}+\theta_3C_{28}T\log q,$$

bu yerda

$$C_{28}=\frac{1}{2\pi}\left(1+\frac{1}{T_0\log 2}\right)+\frac{C_{26}}{T_0}+\frac{C_{26}}{\log q_0}\max_{\mathrm{T}\geq T_0}\frac{\log T}{T} \quad (3.10)$$

(3.8) – (3.10) ifodalarda $T_0=14$ va $q_0=3$ deb faraz qilsak, 3.2 teoremaning tasdig‘iga ega bo‘lamiz.

III.3-§. $\psi(x,\chi)$ funksiya uchun aniqlashtirilgan formula

Biz bu paragrafda $[x-h,x]$ intervaldagi tub sonlar soni uchun mavjud baholarni aniqlashtiramiz. Avvalo

$$\psi(x,\chi)=\sum_{n\leq x}\chi(n)\Lambda(n)$$

funksiya[9,12] uchun formuladagi qoldiq hadda qatnashuvchi o‘zgarmasning son qiymatini aniqlaymiz. Bu yerda quyidagi teorema o‘rinli.

3.1-teorema. Agar $\chi(modq)$ ixtiyoriy xarakter va $3\leq T\leq x$ bo‘lsa, u holda

$$\psi(x,\chi) = \delta_\chi x - \mathrm{E}_{\tilde{\beta}} \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + R(x,T), \qquad (3.11)$$

bunda

$$\delta_\chi = \begin{cases} 1, \text{agar } \chi = \chi_0 - \text{bosh xarakter bo'lsa;} \\ 0, \qquad \text{agarda } \chi \neq \chi_0 \text{bo'lsa,} \end{cases}$$

o‘ng tomondagi yig‘indi $L-$ funksiya $L(s,\chi)$ning $0 < \sigma < 1, |t| \leq T$ to‘g‘ri to‘rtburchakdagi maxsus haqiqiy nollaridan boshqa barcha nollari $\rho = \beta + i\gamma$ bo‘yicha olinadi va

$$|R(x,T)| < 1445.9163 \frac{x}{T} log^2 qx + \mathrm{E}_{\tilde{\beta}} x^{\frac{1}{4}} logx$$
$$+ 2{,}6721(logx) \min\left(1, \frac{x}{\pi < x > T}\right). \qquad (3.12)$$

(3.12) da $< x >$ bilan x dan unga eng yaqin turgan butun songacha bo‘lgan masofa belgilangan.

Quyidagi belgilashlarni kiritamiz: $\delta(y)$ bilan

$$\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} y^s \frac{ds}{s} = \begin{cases} 0, \text{agar } 0 < y < 1 \text{ bo'lsa;} \\ \frac{1}{2}, \text{ agar } y = 1 \text{ bo'lsa;} \quad (\varkappa > 0) \\ 1, \text{agarda } y > 1 \text{ bo‘lsa} \end{cases}$$

tenglikning o‘ng tomonini belgilaymiz va

$$I(y,T) = \frac{1}{2\pi i}\int_{\varkappa - iT}^{\varkappa + iT} y^s \frac{ds}{s}$$

bo‘lsin. Teoremaning isbotida biz quyidagi lemmalardan foydalanamiz.

3.3-lemma. Agar $y > 0$, $\varkappa > 0$, va $T > 0$ bo‘lsa, quyidagi tengsizlik o‘rinli bo‘ladi:

$$|I(y,T) - \delta(y)| < \begin{cases} y^\varkappa \, min(1, \quad (\pi T)^{-1}|logy|^{-1}), \text{agar } y \neq 1 \text{ bo'lsa;} \\ \varkappa(\pi T)^{-1}, \text{ agarda } y = 1 \text{ bo'lsa.} \end{cases}$$

Bu lemmaning isboti [9] ning 17-§ da keltirilgan.

Endi $\chi(modq)$ ixtiyoriy primitiv xarakter bo‘lsin. $\sigma \leq -1$ yarim tekislikdan $|s + a + 2m| \leq \frac{1}{4}$, $a = \{1 - \chi(-1)\}$, $m = 0, 1, 2, ...$ ko‘rinishdagi doirada yotuvchi nuqtalarni chiqarib tashlaymiz va qolgan sohani G bilan belgilaymiz.

3.4-lemma. Agar $q \geq q_0$ va $|t| \geq t_0$ bo‘lsa, G sohada

$$\left|\frac{L'}{L}(s,\chi)\right| \leq c_1 \ln(q|s|)$$

tengsizlik o‘rinli. Bunda

$$c_1 = \left(1 + \frac{2{,}5708}{\sqrt{4 + t_0^2 \ln(4 + t_0^2)}}\right) \cdot \left(1 + \frac{2\pi}{ln(4 + t_0^2)}\right) \cdot \left(1 + \frac{ln\left(1 + \frac{1}{\sqrt{1 + t_0^2}}\right)}{ln\, q_0 \sqrt{1 + t_0^2}}\right)$$

$$+ \frac{6{,}2759}{ln\, q_0 \sqrt{1 + t_0^2}}.$$

3.5-lemma. Agar $\chi(modq)$ ixtiyoriy primitiv xarakter va $|s| \geq |s_0| > 2$ bo‘lsa,

$$\sum_{\rho} |2 - \rho|^{-2} \leq c_2\, lnq$$

tengsizlik o‘rinli.

Bu yerda yig‘indi Dirixle $L-$ funksiya $L(s,\chi)$ning barcha trivial bo‘lmagan nollari $\rho = \beta + i\gamma$ bo‘yicha olinadi va

$$c_2 = 48{,}6882 + 6{,}5834 \left\{ \left(\frac{ln \frac{2}{\sqrt[4]{\pi}}}{|s_0| ln |s_0|} + |s_0|^{-1} + c_3 \right) (1 - 2|s_0|^{-1})^{-1} [1 + \right.$$

$$\left. (\ln(1 - 2|s_0|^{-1})^{-1})\, (lnR_0)^{-1}] \right\}, \quad R \geq R_0 \geq 3,\ |s| \geq |s_0| > 2$$

$$c_3 = \frac{1}{2}\left(1 + \frac{a}{|s_0|}\right)\left(1 + |s_0 + a|^{-1} + \frac{1}{ln \frac{|s_0 + a|}{2}} + \frac{ln 2\pi}{|s_0 + a| ln \frac{|s_0 + a|}{2}}\right.$$

$$\left. + \frac{2\left|r\left(\frac{1}{s + a}\right)\right|}{|s_0 + a| ln \frac{|s_0 + a|}{2}}\right)$$

va

$$\left|r\left(\frac{1}{s + a}\right)\right| \leq \frac{1}{12|s_0|}\left(1 + \frac{1}{30|s_0|^2} + \frac{1}{105|s_0|^4}\right),$$

$$|s| \geq |s_0| > 2$$

Lemmaning isboti [16] da keltirilgan.

Teoremaning isboti. $\chi(modq)$ ixtiyoriy xarakter bo‘lsin. $\psi_0(x,\chi)$ ni

$$\psi_0(x,\chi)=\sum\nolimits'_{n\le x}\chi(n)\Lambda(n)=\begin{cases}\psi(x,\chi), & \text{agar } x\neq p^{\alpha}\text{bo'lsa;}\\ \psi(x,\chi)-\dfrac{1}{2}\Lambda(x), & \text{agar } x=p^{\alpha}\text{bo'lsa}\end{cases}$$

tenglik bilan aniqlab 3.3- lemmani qo‘llaymiz, u holda quyidagilarga ega bo‘lamiz:

$$|\psi_0(x,\chi)-J(x,\chi,T)|<\sum_{\substack{n=1\\ n\neq x}}^{\infty}\Lambda(n)\left(\frac{x}{n}\right)^{\varkappa}min\left(1,\frac{(\pi T)^{-1}}{\left|ln\frac{x}{n}\right|}\right)+\varkappa\frac{\Lambda(n)}{\pi T}\ , \qquad (3.13)$$

bu yerda $\varkappa>1$ va

$$J(x,\chi,T)=\frac{1}{2\pi i}\int_{\varkappa-iT}^{\varkappa+iT}\left\{-\frac{L'}{L}(s,\chi)\right\}\frac{x^s}{s}ds. \qquad (3.14)$$

$\varkappa=1+\ (lnx)^{-1}$ deb olib (3.13) ning o‘ng tomonida turgan qatorni baholaymiz. Avvalo $n\le\frac{3}{4}x$ va $\frac{5}{4}x\le n$ bajariladigan hadlarni baholaymiz. [16] dagi I.1-lemmaga asosan $1<\sigma<1{,}03$ bo‘lsa,

$$\frac{\zeta'}{\zeta}(\sigma)<\frac{1}{\sigma-1}-\gamma_0+0{,}1857(\sigma-1)<\frac{1}{\sigma-1}$$

bajariladi. Bu yerda $\zeta(s)$ Riman funksiyasi bo‘lib $Res=Re(\sigma+it)=\sigma>1$ bo‘lganda

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

tenglik bilan aniqlanadi. Bu tengsizlik va $x^{\varkappa}=ex$ ekanligidan (3.13) ning o‘ng tomonidagi yig‘indida $n\le\frac{3}{4}x$ shartni qanoatlantiruvchi hadlarning xissasi

$$\frac{ex}{\pi Tlog\frac{4}{3}}\left\{-\frac{\zeta'}{\zeta}(\varkappa)\right\}<\frac{e}{\pi log\frac{4}{3}}\cdot\frac{x}{T}\cdot logx, x\ge x_0\ .$$

dan ko‘p emas. Shunga o‘xshash $\frac{5}{4}x\le n$ shartni qanoatlantiruvchi hadlarning hissasi

$$\frac{e}{\pi log\frac{5}{4}}\cdot\frac{x}{T}logx\ ,\quad x\ge x_0$$

dan ko‘p emas. Endi $\frac{3}{4}x < n < x$ tengsizlik o‘rinli bo‘lgan hadlarni qaraymiz.$x_1$bilan x dan kichik tub sonning eng katta darajasini belgilaymiz, ya’ni shartni qanoatlantiruvchi $x_1 = p^{\alpha} < x$eng katta son(agar $x = p^{\alpha}$bo‘lsa,$x_1 \neq x$). Biz$\frac{3}{4}x < x_1 < x$deb hisoblashimiz mumkin.$n = x_1$ bo‘lsa, $log\frac{x}{n} \geq \frac{x-x_1}{x}$ bo‘ladi. Shuning uchun ham $n = x_1$ da (3.13) dan

$$\Lambda(x_1)\left(\frac{x}{x_1}\right)^{\varkappa} min\left(1,\frac{x}{\pi T|x-x_1|}\right) < \left(\frac{4}{3}\right)^{\varkappa_0}(lnx)min\left(1,\frac{xT^{-1}}{\pi|x-x_1|}\right),$$

bunda $\varkappa_0 = 1 + (lnx_0)^{-1}$, $x \geq x_0$.

Qolgan hadlar uchun $n = x_1 - \nu$, $0 < \nu < \frac{x}{4}$ deb olamiz. U holda $ln\frac{x}{n} \geq \frac{\nu}{x_1}$. Shuning uchun ham bu hadlarning (3.13) ning o‘ng tomonidagi yig‘indiga qo‘shadigan hissasi

$$\left(\frac{4}{3}\right)^{\varkappa}\frac{x}{\pi T}(lnx)\sum_{0<\nu<\frac{x}{4}}\nu^{-1} <$$

$$< \frac{1}{\pi}\left(\frac{4}{3}\right)^{\varkappa_0}(1 + 2x_0^{-1}(lnx_0)^{-1} + x_0^{-2}(3lnx_0)^{-1})xT^{-1}log^2x$$

dan ko‘p emas[22]. Endi (3.13) formulaningo‘ngtomonidagi$n$ ning $x < n < \frac{5}{4}x$ qiymatlariga mos hadlarni baholaymiz.$x_2$bilan x dan katta tub sonning eng kichik darajasini belgilaymiz, ya’ni shartni qanoatlantiruvchi $x_1 = p^{\alpha} > x$eng kichik son bo‘lsin. U holda $\frac{3}{4}x < n < x$ bo‘lgan yuqorida qaralgan singari mulohaza yuritib qaralayotgan hadlarning (3.13) ning o‘ng tomonidagi hissasi $x \geq x_0$ bo‘lganda

$$\left(1 + \frac{0{,}2232}{lnx_0}\right)(lnx)min\left(1,\frac{xT^{-1}}{\pi T|x-x_2|}\right) +$$

$$+\frac{1}{\pi}\left(1 + \frac{0{,}2232}{lnx_0}\right)\left(1 + \frac{(2 + (3x_0)^{-1})}{x_0 lnx_0}\right)\frac{x}{T}\ ln^2x$$

dan ortiq emas degan hulosaga kelamiz. Olingan barcha baholarni yig‘ib (3.13) dan

$$|\psi_0(x,\chi) - J(x,\chi,T)| < M_1\frac{x}{T}\ ln^2x + M_2(lnx)\min\left(1,\frac{x}{\pi < x > T}\right) \quad (3.15)$$

bunda

$$M_2 = 1 + 0{,}2232(lnx_0)^{-1} + \left(\frac{4}{3}\right)^{\varkappa_0},$$

$$M_1 = \frac{1}{\pi}\left\{\left(\left(\frac{4}{3}\right)^{\varkappa_0} + 1 + \frac{0{,}2232}{lnx_0}\right)\left(1 + \frac{(2+(3x_0)^{-1})}{x_0 lnx_0}\right) + (8{,}0506e + x_0^{-1} + \right.$$

$$\left. +(x_0 lnx_0)^{-1})(lnx_0)^{-1}\right\}.$$

Endi (3.14) dagi integrallashning vertikal yo‘lini uchlari $\varkappa \pm iT, -U \pm iT$ nuqtalarda bo‘lgan to‘g‘ri to‘rtburchakning qolgan uchta tomoni bo‘yicha olingan integral bilan almashtiramiz. Bunda $U -$ yetarlicha katta $a = 0$ bo‘lsa toq, agarda $a = 1$ bo‘lsa juft son. Shunday qilib vertikal tomon $L(s,\chi)$ ning ikkita trivial nolining o‘rtasidan o‘tadi. Integral ostidagi funksiyaning to‘g‘ri to‘rtburchak ichidagi qutblaridagi chegirmalarining yig‘indisi

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - (1-a)lnx - b(\chi) + \sum_{m\le U} \frac{x^{a-2m}}{2m-a} \tag{3.16}$$

ga teng bo‘ladi. $-1 \le \sigma \le 2$ bo‘lganda ixtiyoriy q, T lar uchun $L(s,\chi)$ funksiyaning $|T-\gamma| < 1$ shartni qanoatlantiruvchi nollari soni [12] dagi (2.12) formulaga asosan $\theta_1 c_4 logqT$ ga teng va bu nollarning ordinatalari orasida uzunligi $c_4^{-1}(logqT)^{-1}$ dan kam bo‘lmagan masofa bo‘lishi kerak. Bu yerda

$$c_4 = 5\left\{\frac{1}{2}\left(1 + \left(\gamma_0 + \frac{1}{2T_0} + \frac{1}{12T_0^2} + \frac{1}{2}\sqrt{1+\frac{9}{T^2}}\right)(lnT_0)^{-1}\right)\right.$$

$$\left. + 0{,}03486(lnq_0T_0)^{-1}\right\}, \quad q \ge q_0 \ge 3, T \ge T_0 \ge 2.$$

T ni $|T-\gamma| \ge (c_4 logqT)^{-1}$ bajariladigan qilib tanlaymiz ([12] ning 258 betiga qarang) , u holda $-1 \le \sigma \le 2$, $s = \sigma + iT$, $T \ge T_0(\ge 3)$,

$$\left|\frac{L'}{L}(s,\chi)\right| \le c_5 logqT,$$

bunda

$$c_5 = c_4^2 + \left\{10\left(1 + \left(\gamma_0 + \frac{1}{2}T_0^{-1} + \frac{1}{12}T_0^{-2} + \frac{1}{2}\sqrt{1 + 9T_0^{-2}}\right)(lnT_0)^{-1}\right)\right.$$

$$\left. + 3{,}5629(lnq_0T_0)^{-1}\right\}(lnq_0T_0)^{-1}.$$

Shuning uchun ham σ ning bu qiymatlarida gorizontal kesma bo‘yicha olingan integral

$$c_5 < c_5T^{-1}(ln^2qT)\int_{-\infty}^{\varkappa} x^{\sigma}d\sigma = e\,c_5x(ln^2qT)(Tlnx)^{-1} \qquad (3.17)$$

dan katta emas.

Endi $-U \leq \sigma \leq 1$ gorizontal kesma va $\sigma = -U$ vertikal to‘g‘ri chiziq bo‘yicha olingan integrallarni baholash qoldi. 3.4-lemmaga asosan gorizontal kesmaning qolgan qismi bo‘yicha colingan integral uchun

$$c_1\int_{-U}^{-1}\frac{lnq|s|}{|s|}x^{\sigma}d\sigma \leq c_1\frac{l}{T}\int_{-\infty}^{-1}x^{\sigma}d\sigma = c_1\frac{lnqT}{xTlnx} \qquad (3.18)$$

baho o‘rinli. Vertikal to‘g‘ri chiziq bo‘yicha olingan integral esa $U \to \infty$ da nolga intiladi. shunday qilib $(3.15) - (3.18)$ lar va

$$\sum_{m=1}^{\infty}\frac{x^{-(2m-a)}}{2m-a} < \sum_{n=1}^{\infty}x^{-n} = \frac{1}{x-1}, \quad (x > 1)$$

dan

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} - b(\chi) + \theta_2M_3\frac{xln^2qx}{T}$$

$$+ \theta_3M_2(lnx)\min\left(1, \frac{x}{\pi < x > T}\right), \qquad (3.19)$$

ni hosil qilamiz. Bu yerdagi $|\theta_i| \leq 1, i = 1, 2, 3$;

$$M_3 = \frac{ec_5}{\pi lnx_0} + \frac{c_1}{\pi x_0^2(lnq_0T_0)lnx_0} + \frac{(lnq_0T_0)^{-2}}{x_0 - 1} + \frac{1}{lnx_0} + M_1\ ;$$

M_1 va M_2 lar esa yuqoridagi (3.15) bahoda aniqlangan.

Endi $b(\chi)$ ni boshqacharoq ko‘einishda tasvirlab olamiz. [9]ning 12-§dagi (17) formulaga ko‘ra

$$\frac{L'}{L}(s,\chi) = -\frac{1}{2}log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \qquad (3.20)$$

Bu yerda $s = 2$ deb olib hosil bo‘lgan tenglikni (3.20) dan ayirsak

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(2,\chi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{a}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2-\rho}\right) \quad (3.21)$$

hosil bo‘ladi. Bu yerdan $a = \frac{1}{2}\{1-\chi(-1)\} = 1$ bo‘lganda

$$b(\chi) \equiv \frac{L'}{L}(0,\chi) = 1{,}64585\theta_4 - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \qquad (3.22)$$

ni hosil qilamiz. Chunki

$$\left|\frac{L'}{L}(2,\chi)\right| < 0{,}64585,$$

$$-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) = \gamma_0 + 2log2$$

$$\frac{\Gamma'}{\Gamma}\left(1+\frac{a}{2}\right) = \begin{cases} -\gamma_0, & a = 0; \\ -\gamma_0 + 2 - log2, & a = 1. \end{cases}$$

$a = 0$ bo‘lsa, ushbu yoyilma

$$\frac{L'}{L}(s,\chi) = s^{-1} + b(\chi) + sb_1 + \cdots$$

ni qaraymiz. (3.21) ga asosan va

$$-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) = \frac{\gamma_0}{2} + \frac{1}{s+a} + \sum_{n=1}^{\infty}\left(\frac{1}{s+a+2n} - \frac{1}{2n}\right)$$

dan

$$b(\chi) = 0{,}64585\theta_5 - \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{2-\rho}\right), \quad |\theta_5| < 1. \qquad (3.23)$$

ni hosil qilamiz. (3.22) va (3.23) lardan $0 < \theta_4 < 1$ ni mos qilib tanlaganimizda doimo (3.22) ning bajarilishiga ishonch hosil qilamiz.

(3.22) ning o‘g tomonidagi qatorni qaraymiz. Bu qatorning $|\gamma| \geq 1$shartni qanoatlantiruvchi hadlari uchun

$$\sum_{|\gamma|\geq 1}\left|\frac{1}{\rho}+\frac{1}{2-\rho}\right| = 2\sum_{|\gamma|\geq 1}\frac{1}{|\rho(2-\rho)|}\leq 6\sum_{|\gamma|\geq 1}|2-\rho|^{-2}$$

tengsizlik o‘rinli. Agarda $|\gamma| < 1$ bo‘lsa, u holda

$$\sum_{|\gamma|<1}|2-\rho|^{-1}\leq\sqrt{5}\sum_{|\gamma|<1}|2-\rho|^{-2}.$$

Shuning uchun ham (3.22) ga 3.5-lemmani qo‘llab

$$b(\chi) = c_6\theta_6 lnq\sum_{|\gamma|<1}\frac{1}{\rho}$$

ni hosil qilamiz. Bunda

$$c_6 = \left(6+\sqrt{5}\right)c_2 + 1{,}64585(lnq_0)^{-1}.$$

Bundan foydalanib (3.19) ni quyidagicha yozish mumkin:

$$\psi_0(x,\chi) = -\sum\nolimits'_{|\gamma|<T}\frac{x^{\rho}}{\rho} - -\sum\nolimits'_{|\gamma|<1}\frac{1}{\rho} - \frac{x^{1-\tilde{\beta}}-1}{1-\tilde{\beta}} - \frac{x^{\tilde{\beta}}-1}{\tilde{\beta}} + R_1(x,T),$$

Bunda $\sum'-$ maxsus nollar $\tilde{\beta}$ va $1-\tilde{\beta}$ dan boshqa barcha nollar bo‘yicha olingan yig‘indini bildiradi va

$$R_1(x,T) < \left(M_3 + \frac{c_6}{lnq_0x_0}\right)\frac{x}{T}ln^2qx + M_2(lnx)\min\left(1,\frac{x}{\pi < x > T}\right).$$

$\frac{3}{4}\leq\tilde{\beta} < 1$ bo‘lganligi sababli $\tilde{\beta}^{-1}\leq\frac{4}{3}$. Shuningdek

$$\left(x^{1-\tilde{\beta}}-1\right)\left(1-\tilde{\beta}\right)^{-1}\leq x^{\frac{1}{4}}lnx.$$

Shu sababli $\left(x^{1-\tilde{\beta}}-1\right)\left(1-\tilde{\beta}\right)^{-1}$ va $\tilde{\beta}^{-1}$ larni qoldiq hadga qo‘shish mumkin. $\sum'\frac{1}{\rho}-$ yig‘indini ham qoldiq hadga qo‘shish mumkin. Haqiqatan ham, $L(s,\chi)$ funksiya [12] I.2 va I.3-lemmalarga asosan

$$\sigma\geq 1-c_7\left(lnq(|t|+3)\right)^{-1},\ t - \text{ixtiyoriy haqiqiy son}$$

sohada nolga aylanmaydi. Bunda $c_7 = 0{,}0109986$.

Bundan $|t|\leq 1$ bo‘lganda $\sigma\geq 1-c_8(logq)^{-1}$ni hosil qilamiz. Bu yerda $c_8 = c_7(1+(logq_0)^{-1}log4)^{-1}$. $L(s,\chi)$ funksiyaning nollari $\sigma=\frac{1}{2}$ to‘g‘ri chiziqqa nisbatan simmetrik joylashgani uchun ular $\beta > c_9(logq)^{-1}$tengsizlikni qanoatlantirishi kerak va $|\gamma| < 1$ shartni qanoatlantiruvchi nollar soni $c_{10}lnq$, (bunda $c_{10} = 35{,}0715$) dan ko‘p emas. shuning uchun ham

$$\left|\sum\nolimits'_{|\gamma|<1}\frac{1}{\rho}\right| \le \sum{}' \frac{1}{\beta} \le c_9 ln^2 q, \quad c_9 = c_{10} c_8^{-1}.$$

Bulardan va $\psi_0(x,\chi)$ ning aniqlanishiga ko‘ra

$$\psi(x,\chi) = -\mathrm{E}_{\tilde{\beta}} \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum\nolimits'_{|\gamma|<T} \frac{x^\rho}{\rho} + R_1(x,T)\,, \tag{3.24}$$

bunda

$$|R_2(x,T)| < \left(M_3 + c_6(lnq_0x_0)^{-1} + \frac{4}{3}(lnq_0x_0)^{-2} + c_9 + (2lnq_0x_0)^{-1}\right)\frac{x}{T} ln^2 qx$$
$$+ M_2(lnx)\min\left(1, \frac{x}{\pi < x > T}\right) + E_{\tilde{\beta}} x^{\frac{1}{4}} lnx\,. \tag{3.25}$$

Hozirgacha biz primitiv xarakterlarni qaradik. $\chi = \chi_0^*(mod 1)$ − bosh xarakter bo‘lsin. U holda $\psi(x,\chi_0^*) = \psi(x)$ bo‘ladi. Shuning uchun ham bu holda yuqorida qaralgan $\psi_0(x,\chi)$ va $L(s,\chi)$ larning o‘rniga mos ravishda $\psi_0(x)$ va $\zeta(s)$ funksiyalarni qarab quyidagiga ega bo‘lamiz:

$$\psi(x,\chi_0^*) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + \bar{R}_2(x,T)\,, \tag{3.26}$$

$$|\bar{R}_2(x,T)|$$
$$< \left\{(lnx_0)^{-1} + (2(x_0^2-1)ln^2x_0)^{-1} + \frac{ln2\pi}{ln^2x_0} + \frac{1}{\pi}\left(\frac{c_{11}}{lnx_0} + \frac{c_{12}x_0^{-2}}{ln^2x_0}\right) + M_1\right\}\frac{x}{T} ln^2 qx$$
$$+ M_2(lnx)\min\left(1, \frac{x}{\pi < x > T}\right). \tag{3.27}$$

bu yerda

$$c_{11} = 25e\left\{(T_0^{-2} + 0{,}64585)(lnT_0)^{-1}\right.$$
$$\left. + \frac{1}{2}\left[1 + \left(\gamma_0 + \frac{1}{2}T_0^{-1} + \frac{1}{12}T_0^{-2} + \frac{1}{2}\sqrt{1+4T_0^{-2}}\right)(lnT_0)^{-1}\right]\right\}^2 +$$
$$+ e(23T_0^{-2} + 0{,}64585)(lnT_0)^{-2} +$$
$$+ 11e\left[1 + \left(\gamma_0 + \frac{1}{2}T_0^{-1} + \frac{1}{12}T_0^{-2} + \frac{1}{2}\sqrt{1+4T_0^{-2}}\right)(lnT_0)^{-1}\right](lnT_0)^{-1};$$

$$c_{12} = \left\{2 + \frac{2}{\sqrt{4+T_0^2}}(ln(4+T_0^2))^{-1} + 1{,}2418\,(ln(1+T_0^2))^{-1}\right\}$$

$$\left(1 + (2lnT_0)^{-1}\, ln(1+T_0^{-2})\right).$$

(3.25) va (3.27) lardan $|\bar{R}_2(x,T)|$ uchun olingan baho$|R_2(x,T)|$ uchun olingan bahodan kichik ekaligini ko‘rish qiyin emas.

Nihoyat $\chi(modq)-$ xarakter $\chi_1(modq_1)-$primitiv xarakter bilan indutsirlangan hosilaviy (primitiv bo‘lmagan) xarakter bo‘lsin. U holda

$$|\psi(x,\chi) - \psi(x,\chi_1)| \le \sum_{\substack{p^v \le x, p\backslash q \\ p\nmid q_1}} lnp \le \frac{lnx}{ln2} \sum_{p\backslash q, p\nmid q_1} lnp < \frac{lnx}{ln2} lnq.$$

Shuning uchun ham (3.24) va(3.25) lardan

$$\psi(x,\chi) = \delta_\chi x - \mathrm{E}_{\tilde{\beta}} \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum\nolimits'_{|\gamma|<T} \frac{x^\rho}{\rho} + R_3(x,T)$$

ga ega bo‘lamiz. Bunda

$$|R_3(x,T)| < \left\{\frac{ec_5}{\pi lnx_0} + \frac{c_1}{\pi x_0^2(lnq_0T_0)lnx_0} + \frac{(lnq_0T_0)^{-2}}{x_0-1} + \frac{1}{lnx_0} + M_1\right.$$

$$\left. + c_6(lnq_0x_0)^{-1} + \frac{4}{3}(lnq_0x_0)^{-2} + c_9 + (2lnq_0x_0)^{-1} + \frac{1}{ln2}\right\}\frac{x}{T}ln^2qx$$

$$+ M_2(lnx)\min\left(1, \frac{x}{\pi < x > T}\right) + E_{\tilde{\beta}} x^{\frac{1}{4}} lnx\,.$$

Bu yerdan $T_0 = 3,\ q_0 = 3,\ x_0 = 3$ deb 1.1-teoremadagi tasdiqqa kelamiz.

Bu teoremadan quyidagi natija kelib chiqadi:

3.4-natija. Agar $q_0 \le q \le P,\ \ 3 \le T_0 \le T\ \left(= P^{\frac{19}{4}}\right), T^{2c} \le x$ va x butun son bo‘lsa, u holda

$$\psi(x,\chi) = \delta_\chi x - E_{\tilde{\beta}} \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum\nolimits'_{|\gamma|<T} \frac{x^\rho}{\rho} + R(x,T)$$

o‘rinli. Bu yerda

$$|R(x,T)| < \left\{ x_0^{-\eta} \left[\left(\frac{4}{3} + \frac{1}{x_0 - 1} \right) \frac{1}{ln^2 x_0} + \left(\frac{19c}{2} ln2 \right)^{-1} + c_9 \left(\frac{361c^2}{4} \right)^{-1} \right. \right.$$

$$\left. + c_6 \left(\frac{19c}{2} ln x_0 \right)^{-1} \right] + x_0^{-\eta + \frac{1}{4}} (ln x_0)^{-1} + \frac{1}{2\pi} [c_1 (c x_0 ln x_0)^{-2}$$

$$\left. + c_5 e (2c^2 ln x_0)^{-1}] + x_0^{-\frac{1}{2}} + M_1 \right\} \frac{x}{T} ln^2 x, \;\; \eta = 1 - (2c)^{-1}.$$

bu natija teorema isbotining tegishli joylarida natijadagi shartlarni inobatga olsak, teoremadan kelib chiqadi.

Uchinchi bob yuzasidan xulosa

Ushbu bobda L-funksiyaning logarifmik hosilasini nollari bo‘yicha qatorga yoyish ko‘rsatilgan.

$N(T,\chi)$funksiya uchun formulaning qoldiq hadiga kiruvchi o‘zgarmasning qiymatini aniqlangan.

$\psi(x,\chi)$ funksiya uchun aniq baholar olingan.Umuman olganda bunday formulalar mavjud lekin ularning qoldiq hadi ”O”-simvoli ishtirok etgani uchun ba’zi masalalarda qo‘llab bo‘lmaydi. ”O”-simvoli ishtirok etmagan formula isbotlangan.

XULOSA

Ushbu dissertatsiyaning birinchi bobida cheksiz ko'paytmalar funksiyalarni cheksiz qatorlardan tashqari cheksiz ko'paytmalar yordamida ifodalash muhim ahamiyatga ekanligi yoritilgan. Yaqinlashuvchi va uzoqlashuvchi cheksiz ko'paytmalar, cheksiz ko'paytma yaqinlashishining zaruriy va yetarli shartlari yoritib berilgan. Cheksiz ko'paytmalarning yaqinlashuvchi bo'lishini tengsizlik yordamida ifodalangan Cheksiz ko'paytmalarga doir misollar yechimi ko'rsatilgan. Chekli tartibli butun funksiyalar, Veyershtrass formulasi haqida ma'lumot berilgan. Chekli tartibli butun funksiyalarni cheksiz ko'paytmalar yordamida ifodalash ko'rsatilgan, misollar keltirilgan.

Ushbu dissertatsiyaning ikkinchi bobi $\zeta(s)$-funksiyaning logarifmik hosilasini nollari bo'yicha qatorga yoyish. $\zeta(s)$-funksiyaning nollari haqida teoremalar keltirilgan. $N(T)$funksiya uchun formulaning qoldiq hadiga kiruvchi o'zgarmasning qiymati aniqlangan. $\psi(x)$ funksiya uchun baholar olingan.

Faraz etaylik q ixtiyoriy natural son, p −tub son, $s=\sigma+it$ −kompleks son, χ esa $q$moduli bo'yicha Dirixle xarakteri, $\Lambda(n)$ bilan

$$\Lambda(n)=\begin{cases}\ln p, & \text{agar } n=p^{\alpha} \text{ bo'lsa,}\\ 0, & \text{agar } n\neq p^{\alpha} \text{ bo'lsa}\end{cases}$$

tenglik bilan aniqlanuvchi Mangoldt funksiyasi bo'lsin. Ma'lumki, Dirixle $L-$ funksiyasi $L(s,\chi)$, Re $s=\sigma>1$ bo'lganda

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

tenglik bilan aniqlanadi. Bu funksiyaning tadbiqlagida $\psi(x,\chi)=\sum_{n\le x}\chi(n)\Lambda(n)$ funksiya uchun aniq formula muhim ahamiyatga ega. Bunday formulalar umuman olganda mavjud, lekin ularning qoldiq hadida "O"-simvoli ishtiroq etgani uchun ba'zi bir sonli hisoblashlar qatnashgan masalalarda foydalanib bo'lmaydi. Ushbu ishda biz $\psi(x,\chi)$ funksiya uchun "O"-simvoli ishtiroq etmagani quyidagi natijani isbotlaymiz.

Teorema. Agar χ q moduli bo'yicha Dirixle xarakteri va $3\le T\le x$ bo'lsa. U holda

$$\psi(x,\chi)=\delta_{\chi}x-E_{\widetilde{\beta}}\frac{x^{\widetilde{\beta}}}{\widetilde{\beta}}-\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho}+R(x,T), \quad (1)$$

bu yerda $\delta_\chi \chi \neq \chi_0$ yoki $\chi = \chi_0$ (χ_0 $-$ bosh xarakter) bo‘lishiga qarab 0 yoki 1

$$\delta_\chi = \begin{cases} 1, & \text{agar } \chi = \chi_0 - \text{bosh xarakter bo‘lsa,} \\ 0, & \text{agar } \chi \neq \chi_0 - \text{bo‘lsa} \end{cases},$$

$$E_{\tilde{\beta}} = \begin{cases} 1, & \text{agar } \chi = \tilde{\chi} - \text{maxsus haqiqiy xarakter bo‘lsa,} \\ 0, & \text{agar } \chi \neq \tilde{\chi} - \text{bo‘lsa} \end{cases},$$

$\tilde{\beta}$ $-$maxsus $\chi = \tilde{\chi}$ $-$haqiqiy xarakterga mos haqiqiy nol bo‘lib o‘ng tomondagi yig‘indi $0 < \sigma < 1, |\gamma| < T$ sohadagi maxsus noldan tashqari barcha $\rho = \beta + i\gamma$ nollar bo‘yicha olinadi. (1) dagi qoldiq had uchun quyidagi baho o‘rinli:

$$|R(x,T)| < 1445{,}91 \frac{x}{T} \log^2 qx + E_{\tilde{\beta}} x^{\frac{1}{4}} \log x +$$

$$+2{,}67(\log x) \min\left(1, \frac{x}{\pi < x > T}\right) \quad (2)$$

bu yerda $< x >$ bilan x dan unga eng yaqin tub sonning darajasigacha bo‘lgan masofa belgilangan.

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