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KIRISH

Magistrlik dissertatsiyasi mavzusining asoslanishi va uning dolzarbligi. O‘zbekiston Resublikasi mustaqillika erishgandan so‘ng ko‘pgina sohalar bilan bir qatorda ta‘lim sohasida ham katta o‘zgarishlar yuz berdi. Bunda asosan rivojlangan davlatlarning ta‘lim tizimlari namuna sifatida olinib, bizning xalqimiz imkoniyatlariva yutuqlariga mos holda yangi ta‘lim tizimi ishlab chiqildi. Hozirgi kunda ilm – fanga prezidentimiz tomonidan alohida e‘tibor berilmoqda [1-4]. Jamiyat ijtimoiy sohasining eng muhim tarkibiy qismlaridan biri ta‘lim tarbiya sohasi bo‘lib, uning rivoji siyosiy–huquqiy, iqtisodiy va ma‘naviy sohalarga bevosita tasir etadi hamdaijtimoiy sohalar me‘yoriy mohiyatini, kamolot darajasini belgilab beradi.

Ta‘lim va fan sohasini rivojlantirish davlat siyosati ma‘no mazmunidan va uning dolzarbligidan kelib chiqib,uni quyidagicha izohlash mumkin:

Birinchiidan, yangi ta‘lim tizimi, barkamol avlod kadrlarini tayyorlashdagi o‘zgarishlar va yangicha yondashuvlar zamonaviy kasb sohalarining paydo bo‘lgani hamda uning mamlakatimiz sharoiti bilan bog‘liqligidir;

Ikkinchiidan, ta‘lim tushunchasi ijtimoiy-iqtisodiy taraqqiyot natijasida muayyan davrdan boshlab, inson faoliyatining alohida mustaqil sohasiga aylanib, jamiyatning ijtimoiy sohasini keyingi bosqichga uzatadi;

Uchinchiidan, ta‘lim inson shaxsining intellektual-ma‘naviy qirralarini shakllantirish, uning jamiyat ishlab chiqarishi va ijtimoiy, siyosiy, madaniy, ma‘rifiy hayotida faol va muvaffaqiyatli ishtirokini ta‘minlashga qaratilgan harakterlar yig‘indisi bo‘lib, ma‘rifiy hamda bilim berishni anglatadi;

To‘rtinchiidan, fan jamiyatning ijtimoiy institutlaridan biri bo‘lib, tabiat va jamiyat hayotini aks ettiruvchi ijtimoiy ong shakli. U katta ilmiy salohiyatini, ijodiy kuch-quvvatini birlashtirib, ma‘naviy barkamol insonni tarbiyalashga, mamlakatda qudratli ilmiy salohiyatni yaratishga xizmat qiladi.

Ushbu magistrlik dissertatsiyasi mavzusi ana shu talab va vazifalardan kelib chiqib tanlandi. Ma‘lumki, sonlar nazariyasining additiv masalalarini tekshirishda trigonometrik yig‘indilarni baholash muhim ahamiyatga ega. Bunday

additiv masalalarga misol sifatida XVIII-asrning o'rtalaridan buyon ma'lum bo'lgan Varing, Eylar-Goldbax, Xardi-Littlvud, Xua-Lo-Ken problemalarini ko'rsatish mumkin. Birinchi bo'lib trigonometrik yig'indilarning trivial bo'lmagan bahosini Veyl va Vinogradovlar olgan [5-8]. Biz bu yerda yuqorida keltirilgan muammolardan ikkitasi:

Varing va Goldbax muammolariga qisqacha to'xtalib o'tamiz.

Varing muammosi. Fransuz matematigi J.L.Lagranj (Giuseppe Lodovico Lagrangia (1736-1813)) 1770 yilda quyidagi teoremani isbotlaydi: "Har qanday natural sonni to'rtta butun son kvadratlari yig'indisi ko'rinishida ifodalash mumkin". Bu teoremadagi tasdiq birinchi bo'lib Diofantning "Arifmetika" sida keltirilgan [9]. Uning isbotini Lagranj qilgan [10].

1770-yilda ingliz matematigi Varing (Edward Waring (1734-1798)) to'rt kvadratlar yig'indisi haqidagi J.L.Lagranj teoremasini umumlashtirishni taklif qildi va bu keyinchalik Varing muammosi deb nomlandi. Zamonaviy terminologiyada, Varingning tasdiqlarida berilgan $k \geq 2$ soni uchun shunday natural son $s = s(k)$ mavjudki, har bir natural son n , $s = s(k)$ ta manfiy bo'lmagan butun sonlarning yig'indisidan iborat bo'ladi, ya'ni n ni

$$n = n_1^k + n_2^k + \dots + n_s^k \quad (1)$$

korinishda ifodalash mumkin. Berilgan k uchun (1) tenglikni qanoatlantiruvchi chekli s ning mavjudligi 1909-yilda nemis matematigi Gilbert (David Hilbert (1862-1943)) tomonidan isbotlangan.

Goldbax muammosi. Nemis matematigi Xristian Goldbax (*Christian Goldbach* (1690-1764) nomi bilan yuritiladigan Goldbax muammosi birinchi marta 1742-yilda Goldbax va Shvetsariyalik nemis matematigi Eylar (*Leonhard Euler* 1707-1783) o'rtasida yozishmalarda paydo bo'lgan. 1742 yilda Xristian Goldbax Leonard Eylarga yozgan xatida quyidagi taxmini aytadi: har bir 5 dan katta toq sonni 3 ta tub sonning yig'indisi ko'rinishida ifodalash mumkin.

Eylar bu masala bilan qiziqib bundanda kuchliroq taxmini ilgari suradi: har bir 2 dan katta juft sonni 2 ta tub sonning yig'indisi ko'rinishida ifodalash mumkin.

Birinchi tasdiq Goldbaxning ternar problemi, ikkinchisi esa Goldbaxning binar problemi (yoki Eyer problemi) deb ataladi [11].

Trigonometrik yig'indilarning umumiy ko'rinishi $S = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)}$ dan

iborat bo'lib, $f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_1 x + \alpha_0$ haqiqiy koeffitsiyentli ko'phad, Q, P -lar natural sonlar. Uning trivial bahosi $|S| \leq P$ dan iborat. Bu trigonometrik yig'indining trivial bo'lmagan bahosi muhim ahamiyatga ega. Bu sohadagi izlanishlar davom etmoqda, chunki yuqorida sanab o'tilgan masalalarning birortasi ham to'la o'z yechimini topmagan. Shuning uchun ham bu sohadagi izlanishlar dolzarb hisoblanadi.

Tadqiqotning ob'yekti va predmeti. Ushbu tadqiqot ishining ob'yekti, trigonometrik yig'indilar, algebraik metodlar, Eyer funksiyasi, Myobus funksiyasi va sonlar nazariyasining ba'zi additiv masalalari, ularni yechish metodlari, olingan baholar tahlili hisoblanadi. Tadqiqotning predmeti bo'lib, sonlar nazariyasining bir nechta additiv masalalarini trigonometrik yig'indilar orqali tatbiqi, olingan baholar va bunda olingan natijalar tahlili.

Ishning maqsadi va vazifalari. Magistrlik dissertatsiyasining asosiy maqsadi

trigonometrik yig'indilarni sonlar nazariyasining ba'zi additiv masalalarini yechishga tatbiq qilishdan iboratdir.

Bu maqsadga erishish uchun quyidagi vazifalarni amalga oshirish kerak:

- Birinchi darajali va ikkinchi darajali trigonometrik yig'indilarni baholash ;
- Trigonometrik yig'indilarning berilgan oraliqdagi n -darajali chegirmalar sonini aniqlashga tatbiq qilish;
- Effektiv baholardagi o'zgarmaslarning boshqa parametrlar bilan bog'liklik ifodasini keltirib chiqarish;
- Sonli baholar olish.

Tadqiqotning ilmiy yangiligi. Sonlar nazariyasida g - eng kichik musbat boshlang'ich ildizni yuqoridan baholash muhim ahamiyatga ega.

B. I. Segallarning va I. M. Vinogradov ishlarida bunday baho mavjud. Ular yetarlicha katta p tub soni va ixtiyoriy $\varepsilon > 0$ son uchun $g = O\left(p^{\frac{1}{2}+\varepsilon}\right)$ ekanligini isbotlaganlar. g uchun olingan bahoda “ O ” -simvoli qatnashganligi uchun undan aniq sonli hisoblashlarda foydalanib bo‘lmaydi. Ushbu ishda g uchun “ O ”-simvoliga bog‘liq bo‘lmagan aniqlashtirilgan $g \leq \left(4^k \sqrt{p} \ln p\right) + 1$ baho isbotlangan. Bu yerda k soni $p-1$ ning tub bo‘luvchilari soni. Olingan natija yuqoridagi shu boradagi natijaning son qiymati jihatdan yaxshilanganidir.

Trigonometrik yig‘indilarni taqqoslamalar sonini topishga va trigonometrik yig‘indilarni berilgan oraliqdagi n -darajali chegirmalar sonini aniqlashga tadbiiq etilgan.

Tadqiqotning asosiy masalalari va farazlari. Tadqiqot ishida quyidagi asosiy masalalar qaraldi:

- n -darajali trigonometrik yig‘indilarni baholash va chegirmalar sonini topishga tatbiiq etish
- Natural sonlar bo‘yicha olingan trigonometrik yig‘indilarning ketma-ketliklar kasr qismlarining taqsimlanishini baholashga tatbiiq
- Tub sonlar bo‘yicha olingan trigonometrik yig‘indilarning additiv masalalarni yechishda tatbiiq
- Sonlar nazariyasining bir qancha yechimini topgan va hali to‘la yechimini topmagan masalalarini o‘rganish oldindan olingan natija va baholarni o‘rganib tahlil qilish.

Tadqiqotning ishlab chiqilgan ilmiy takliflari va amaliy tavsiyalari natijasida sonlar nazariyasining ko‘plab additiv masalalarini trigonometrik yig‘indilarning baholari orqali yechish mumkin bo‘ladi.

Tadqiqot mavzusi bo‘yicha adabiyotlar sharxi. Magistrlik tadqiqot ishini o‘rganish davomida quyidagi adabiyotlardan foydalanildi. I. Allakov [5] , Карацуба А.А. Основы аналитической теории чисел.- М.:Наука,1975.–182 с. Девенпорт Г. Мультипликативная теории чисел.-М: “Наука” 1971г. Чудаков

Н.Г. Введение в теорию функций Дирихле.-М: Гостехиздат.1947г. Pintz J., Elementary methods in the of L-functions v. The theorems of Landau and Page, Acta Arith., 32 (2), 1977, 163-171. Allakov I., Erdanov B.X., Erkinova D.A. Eng kichik musbat boshlang'ich ildizni trigonometrik yig'indilar yordamida yuqoridan baholash. NamDU ilmiy axborotnomasi, 2024. №5. 103-106 b. Allakov I., Erdanov B.X., Erkinova D.A. Application of trigonometric sums to the estimation of the smallest positive initial root. Journal of Theory, Mathematics and Physics. Vol. 3, № 03,2024. 11-16 b. Erkinova D.A. Arifmetik progressiyadagi tub sonlar bo'yicha olingan chiziqli va ikkinchi darajali trigonometrik yig'indilarni baholash. Journal of Theory, Mathematics and Physics. Vol. 3, № 03,2024. 1-6 b. Erkinova D.A. Birinchi va ikkinchi darajali trigonometrik yig'indilarni baholash va ularning tadbiqlariga doir. Algebra va analizning dolzarb masalalari mavzusidagi Respublika ilmiy-amaliy anjumani materiallari to'plami. Termiz-2022. 18-19 noyabr. 25-26 b.

Tadqiqotda qo'llanilgan metodikaning tavsifi. Dissertatsiya ishida ilmiy abstraksiya, tahlil va sintez, monografik kuzatish, taqqoslash, induksiya va deduksiya, statik guruhlash, tizimli tahlil, usullaridan foydalanilgan.

Tadqiqotda olingan natijalarning nazariy va amaliy ahamiyati: Dissertatsiya ishi ilmiy – nazariy xarakterda bo'lib, matematikaning turli additiv masalalarini yechishda, xususan, qo'shiluvchilar soni chegaralangan bo'lgandagi Varing muammosini, Goldbax muammosini, Xarde–Littlvud muammosini, Xua – Lo – Ken muammosini hamda ularning umumlashmalarini yechishda foydalanish mumkin.

Magistrlik dissertatsiya tuzilmasining tavsifi: Dissertatsiya ishi kirish qismi, uchta bob, xulosa va adabiyotlar ro'yxatidan iborat. Kirish qismida mavzuning o'rganilganlik darajasi va dolzarbligi, ishning maqsadi va vazifalari, ishning sinovdan o'tishi, masala tarixi, adabiyotlar tahlili qisqacha berilgan, dissertatsiya ishida qilingan ishlar to'g'risida qisqacha ma'lumotlar berib o'tilgan va asosiy olingan natijalar bayoni keltirilgan.

Birinchi bob trigonometrik yig'indilarning berilgan oraliqdagi n -darajali

chegirmalar sonini aniqlashga tadbiqlari deb ataladi va 3 ta paragrafni o'z ichiga oladi, bu bobda birinchi va ikkinchi darajali trigonometrik yig'indilarni baholari keltirilgan. Bundan tashqari n - darajali trigonometrik yig'indilar baholari olib uni chegirmalar sonini aniqlashga tatbiq etilgan. Bu mavzular yordamchi xarakterga ega, undagi ta'rif va teoremlardan keying bobda foydalaniladi.

Ikkinchi bob natural sonlar bo'yicha olingan trigonometrik yig'indilarning additiv masalalarni yechishga tadbiqlari deb nomlangan va 3 ta paragrafni o'z ichiga oladi, bu bobda Veyl teoremasi, uning isboti va tatbiqlari, Veyl teoremasi tipidagi teoremlarning Varing masalasiga tatbiqlari, ketma ketliklar kasr qismlarining taqsimlanishini baholashda tadbiqlari keltirilgan.

Dissertatsiyaning uchinchi bobi tub sonlar bo'yicha olingan trigonometrik yig'indilarning additiv masalalarni yechishda tadbiqlari deb atalgan va 3 ta paragrafni o'z ichiga oladi, bu bobda Vinogradov teoremasi va uning isboti, tub sonlar bo'yicha olingan birinchi darajali trigonometrik yig'indilarning baholari, va uni additiv masalalarni yechishga tatbiqlari keltirilgan. Dissertatsiyaning umumiy hajmi 90 betdan iborat.

I BOB. TRIGONOMETRIK YIG'INDILARNING BERILGAN ORALIQDAGI n - DARAJALI CHEGIRMALAR SONINI ANIQLASHGA TATBIQL.

1.1-§. Birinchi va ikkinchi darajali trigonometrik yig'indilarni baholash

Ma'lumki, sonlar nazariyasining additiv masalalarini tekshirishda trigonometrik yig'indilarni baholash muhim ahamiyatga ega. Bunday additiv masalalarga misol sifatida XVIII asrning o'rtalaridan buyon ma'lum bo'lgan Varing, Eyler-Goldbax, Xardi-Litllvud, Xua-Lo-Ken problemalarini ko'rsatish mumkin. Birinchi bo'lib trigonometrik yig'indilarning trivial bo'lmagan bahosini Veyl va Vinogradovlar olgan.

Ushbu

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}}$$

yig'indini qaraymiz. Bunda a – butun son, m -natural son. S_1 yig'indidagi $e^{2\pi i \xi} = \cos 2\pi \xi + i \sin 2\pi \xi$ ko'rinishdagi funksiya davriy bo'lib uning davri 1 ga teng, ya'ni $e^{2\pi i(\xi+1)} = e^{2\pi i \xi} \cdot e^{2\pi i} = e^{2\pi i \xi}$

Bundan agar $x \equiv y \pmod{m}$ ya'ni $x = y + mt, t \in \mathbb{Z}$ bo'lsa, u holda

$$e^{2\pi i \frac{ax}{m}} = e^{2\pi i \frac{a(y+mt)}{m}} = e^{2\pi i \frac{ay}{m}} \cdot e^{2\pi i at} = e^{2\pi i \frac{ay}{m}}$$

ekanligi kelib chiqadi. Agarda S_1 yig'indida a soni m ga bo'linsa (a/m), u holda $a = mt$ va

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i tx} = 1 + e^{2\pi i t} + e^{2\pi i (2t)} + \dots + e^{2\pi i (m-1)t} = \underbrace{1 + 1 + \dots + 1}_m = m,$$

bo'ladi. Agarda S_1 yig'indida a soni m ga bo'linmasa ($a \nmid m$), u holda

$$S_1 = 1 + e^{2\pi i \frac{a}{m}} + \dots + e^{2\pi i \frac{a(m-1)}{m}} = \frac{1 - \left(e^{2\pi i \frac{a}{m}} \right)^m}{1 - e^{2\pi i \frac{a}{m}}} = \frac{1 - e^{2\pi i a}}{1 - e^{2\pi i \frac{a}{m}}} = 0.$$

Shunday qilib,

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}} = \begin{cases} m, & \text{agar } a / m, \\ 0, & \text{agar } a \nmid m, \end{cases} \quad (1.1.1)$$

ekanligini isbotladik.

Yuqorida qaralgan trigonometrik yig'indni

$$ax \equiv b \pmod{m} \quad (1.1.2)$$

taqqoslamaning yechimlar soni T ni aniqlashga tatbiq etish mumkin. Buning uchun (1.1.2) dan foydalanib T ni quyidagicha yozib olish mumkin.

$$T = \frac{1}{m} \sum_{x=0}^{m-1} \sum_{y=0}^{m-1} e^{2\pi i \frac{(ax-b)y}{m}}. \quad (1.1.3)$$

(1.1.3) da agar (1.1.2) taqqoslama yechimga ega bo'lsa, $ax - b \equiv 0 \pmod{m}$ ya'ni $(ax - b) / m$, qolgan hollarda $(ax - b) \nmid m$ bajariladi. (1.1.3) da $d = (a, m)$, $a = a_1 \cdot d$, $m = m_1 \cdot d$ deb olib yig'ish tartibini o'zgartirsak quyidagiga ega bo'lamiz:

$$\begin{aligned} T &= \sum_{x=0}^{m-1} e^{-2\pi i \frac{by}{m}} \frac{1}{m_1 d} \sum_{x=0}^{m_1 d - 1} e^{2\pi i \frac{a_1 xy}{m_1}} = \sum_{y=0}^{m-1} e^{-2\pi i \frac{by}{m}} \frac{1}{m_1 d} \times \\ &\times \left(\sum_{x=0}^{m_1 - 1} e^{2\pi i \frac{a_1 xy}{m_1}} + \sum_{x=m_1}^{2m_1 - 1} e^{2\pi i \frac{a_1 xy}{m_1}} + \dots + \sum_{x=(d-1)m_1}^{m_1 d - 1} e^{2\pi i \frac{a_1 xy}{m_1}} \right) = \\ &= \sum_{x=0}^{m-1} e^{-2\pi i \frac{by}{m}} \frac{1}{m_1 d} \sum_{x=0}^{m_1 - 1} e^{2\pi i \frac{a_1 xy}{m_1}} = \sum_{y=0}^{m-1} e^{-2\pi i \frac{by}{m}} \frac{1}{m_1} \sum_{x=0}^{m_1 - 1} e^{2\pi i \frac{a_1 xy}{m_1}}. \end{aligned}$$

Bu yerda $(a_1, m_1) = 1$ bo'lgani uchun (1.1.1) ga ko'ra

$$\frac{1}{m_1} \sum_{x=0}^{m_1 - 1} e^{2\pi i \frac{a_1 xy}{m_1}} = \begin{cases} 1, & \text{agar } y / m_1, \\ 0, & \text{agar } y \nmid m_1 \end{cases}$$

Shuning uchun ham yuqoridagi yig'indida faqat y / m_1 hadlar qoladi, qolgan hadlar nolga teng. Shunday qilib, $y = km_1$ deb olsak, $0 \leq km_1 \leq dm_1 - 1$, ya'ni $0 \leq k \leq d - 1$ va

$$T = \sum_{k=0}^{d-1} e^{-2\pi i \frac{bk m_1}{m}} = \sum_{k=0}^{d-1} e^{-2\pi i \frac{bk}{m_1}} = \begin{cases} d, & \text{agar } b / d, \\ 0, & \text{agar } b \neq d. \end{cases}$$

Bizga ma'lumki, agar $d = (a, m)$ bo'lib b / d bo'lsa, (1.1.2) taqqoslama d ta yechimga ega bo'ladi, agarda $b \neq d$ bo'lsa, (1.1.2) taqqoslama birorta ham yechimga ega emas edi.

Yuqorida qarab chiqilgan usul unchalik ham sodda bo'lmasada, u trigonometrik yig'indilardan foydalanib arifmetik masalalarni yechish mumkin ekanligining yoqqol namunasiidir.

Trigonometrik yig'indilarning ba'zi bir xususiy hollardagina aniq qiymatini topish mumkin. Ko'pchilik hollarda esa uning modulining mumkin qadar aniqroq yuqori chegarasini topish masalasi qo'yiladi. Ana shu chegarani topishni biz bundan keyin trigonometrik yig'indilarni baholash, chegaraning o'ziga esa trigonometrik yig'indining bahosi deb ataladi.

Tushunarliki,

$$S_1 = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)},$$

yig'indi uchun trivial (o'z-o'zidan tushunarli bo'lgan) baho $|S| \leq P$ dan iborat.

Bu yerda $f(x)$ haqiqiy koeffitsiyentli ko'phad.

Sonlar nazariyasining ko'pchilik masalalarini yechishda trigonometrik yig'indilarning trivial bahosidan ko'ra aniqroq bahosini topish talab etiladi.

Endi, S_1 dan ko'ra umumiyroq ushbu

$$\sum_{x=Q+1}^{Q+P} e^{2\pi i \alpha x}, \tag{1.1.4}$$

yig'indini baholashni qaraymiz. Bunda α –ixtiyoriy haqiqiy son, Q -butun son, P - butun musbat son.

Avvalo, ixtiyoriy ξ haqiqiy soni uchun

$$|\sin \pi \xi| \geq 2 \|\xi\|, \tag{1.1.5}$$

ning bajarilishini ko'rsatamiz. Bunda $\|\xi\|$ -bilan ξ dan unga eng yaqin turgan butun songacha bo'lgan masofa belgilangan.

(1.1.5) ning ikkala tomoni ham juft va davriy funksiya bo'lib, davri 1 ga teng. Shuning uchun ham $0 \leq \xi \leq \frac{1}{2}$ bo'lganda $\sin \pi \xi \geq 2\xi$ ning bajarilishini ko'rsatish yetarli. $f(\xi) = \sin \pi \xi - 2\xi$ funksiyaning qaraymiz. Bu funksiya $\xi = 0$ va $\xi = \frac{1}{2}$ da nolga aylanadi. Uning hosilasi $f'(\xi) = \pi \cos \pi \xi - 2$ bo'lib ξ ning faqat bitta qiymatida nolga aylanadi. $f(\xi)$ ning ikkinchi hosilasi $f''(\xi) = -\pi^2 \sin \pi \xi$ bo'lib $0 < \xi < \frac{1}{2}$ da $f''(\xi) < 0$. Demak, $0 \leq \xi \leq \frac{1}{2}$ bo'lganda $f(\xi) \geq 0$ va shuning uchun ham $\sin \pi \xi \geq 2\xi$ bajariladi.

(1.1.5) dan foydalanib, qaralayotgan yig'indini quyidagicha baholash mumkin:

$$\left| \sum_{x=Q+1}^{Q+P} e^{2\pi i \alpha x} \right| = \left| \frac{e^{2\pi i \alpha (Q+P+1)} - e^{2\pi i \alpha (Q+1)}}{e^{2\pi i \alpha} - 1} \right| \leq \frac{2}{|e^{2\pi i \alpha} - 1|} \leq \frac{2}{|e^{\pi i \alpha} - e^{-\pi i \alpha}|} = \frac{1}{|\sin \pi \alpha|} \leq \frac{1}{2\|\alpha\|}.$$

Shunday qilib,

$$\left| \sum_{x=Q+1}^{Q+P} e^{2\pi i \alpha x} \right| \leq \min \left(P; \frac{1}{2\|\alpha\|} \right). \quad (1.1.6)$$

Bundan keyin biz

$$S = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)},$$

yig'indida quyidagi 2 ta holni bir biridan farqlaymiz:

1). $f(x) = \frac{a}{q} x^n$, $(a, q) = 1$ bo'lib S yig'indida x o'zgaruvchi q moduli

bo'yicha chegirmalarning to'la sistemasidagi barcha qiymatlarni qabul qiladi.

Bu holda $x = 0, 1, 2, \dots, q-1$ qiymatlarni qabul qiladi deb qarash mumkin. Shuning uchun ham bu holda

$$S = \sum_{x=0}^{q-1} e^{\frac{2\pi i a x^n}{q}}, (a, q) = 1, \quad (1.1.7)$$

yig'indiga ega bo'lamiz. Bu yig'indiga ratsional trigonometrik yig'indi deb ataladi.

2). $f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \dots + \alpha_{n-1} x + \alpha_n$ bo'lib, $\alpha_0 \neq 0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n$ lar haqiqiy sonlar va $x = Q+1, Q+2, \dots, Q+P$ qiymatlarni qabul qiladi. $Q, P > 0$ lar butun sonlar. Shunday qilib, bu holda

$$S_n = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)},$$

yig'indiga ega bo'lamiz. Bunday yig'indiga H. Veyl yig'indisi deyiladi.

Ikkala holda ham n soniga S_n yig'indining tartibi deyiladi.

Yuqorida biz qaragan S_1 yig'indi 1-tartibli ratsional trigonometrik yig'indi, (1.1.4) esa 1-tartibli Veyl yig'indisi ekan.

Ikkinchi tartibli ratsional trigonometrik yig'indilarni hisoblash. Ushbu

$$S_2 = \sum_{x=0}^{q-1} e^{\frac{2\pi i a x^2}{q}}, (a, q) = 1,$$

yig'indini qaraymiz. Bu yig'indini birinchi bo'lib K.F.Gauss to'la tekshirganligi sababli u Gauss yig'indisi deb ataladi. S_2 yig'indiga qo'shma kompleks bo'lgan yig'indini \bar{S}_2 deb belgilasak, u holda

$$\begin{aligned} |S_2|^2 &= S_2 \cdot \bar{S}_2 = \sum_{x_1=0}^{q-1} e^{\frac{2\pi i a x_1^2}{q}} \cdot \sum_{x_2=0}^{q-1} e^{-\frac{2\pi i a x_2^2}{q}} = \\ &= \sum_{x_1=0}^{q-1} \sum_{x_2=0}^{q-1} e^{\frac{2\pi i a (x_1^2 - x_2^2)}{q}} = \sum_{y_1} \sum_{y_2} e^{\frac{2\pi i a (y_1^2 - y_2^2)}{q}}. \end{aligned}$$

Bu yerda y_1, y_2 lar q moduli bo'yicha chegirmalarning istalgan to'la sistemasini qabul qiladi. $y_1 = y + t$ deb olib t ni y ga bog'liq bo'lmagan holda q moduli bo'yicha chegirmalarning istalgan to'la sistemasini qabul qilsa, $y + t$ (y fiksirlangan bo'lganda) ham shu sistemani qabul qiladi. Shuning uchun ham

$$|S_2|^2 = \sum_{y+t} \sum_y e^{\frac{2\pi i a}{q}((y+t)^2 - y^2)} = \sum_t e^{\frac{2\pi i a}{q}t^2} \cdot \sum_y e^{\frac{2\pi i a}{q}2yt}. \quad (1.1.8)$$

Faraz etaylik avvalo q toq son bo'lsin. U holda $(2a, q) = 1$ va (1.1.1) formulaga asosan

$$\sum_y e^{\frac{2\pi i a}{q}2yt} = \begin{cases} q, & \text{agar } t / q, \\ 0, & \text{agar } t \neq q. \end{cases}$$

Demak, (1.1.8) ning o'ng tomonida faqat $t = 0$ bo'lgan had noldan farqli va

$$|S_2|^2 = e^{\frac{2\pi i a}{q}0^2} \cdot q = q, \text{ ya'ni } |S_2| = \sqrt{q}. \quad (1.1.9)$$

Endi faraz etaylik q juft son bo'lsin. U holda $q = 2q_1$ deb olib (1.1.8) dan quyidagini hosil qilamiz.

$$|S_2|^2 = \sum_t e^{\frac{2\pi i a}{2q_1}t^2} \cdot \sum_y e^{\frac{2\pi i a}{2q_1}2yt} = 2 \sum_t e^{\frac{2\pi i a}{2q_1}t^2} \cdot \sum_{y_1} e^{\frac{2\pi i a}{q_1}y_1t}.$$

Bu yerda y_1 o'zgaruvchi q_1 moduli bo'yicha chegirmalarning to'la sistemasini qabul qiladi. t ning $t = 0, 1, 2, \dots, q_1, q_1 + 1, \dots, 2q_1 - 1 = q - 1$ qiymatlari orasida faqat ikkitasi $t = 0$ va $t = q_1$ lar q_1 ga bo'linadi. Shuning uchun (1.1.1) ga asosan

$$|S_2|^2 = 2 \left(e^{\frac{2\pi i a}{2q_1}0^2} + e^{\frac{2\pi i a}{2q_1}q_1^2} \right) \cdot q_1 = 2 \left(1 + e^{\frac{2\pi i a}{2}q_1} \right) \cdot q_1.$$

Bunda

$$e^{\frac{2\pi i a}{2}q_1} = \begin{cases} 1, & \text{agar } q_1 \text{ juft son bo'lsa,} \\ -1, & \text{agar } q_1 \text{ toq son bo'lsa.} \end{cases}$$

Demak, $|S_2|^2 \leq 4q_1 = 2q$ yoki bundan

$$|S_2| \leq \sqrt{2q} \quad (1.1.10)$$

(1.1.9) va (1.1.10) dan hamma vaqt $|S_2| \leq \sqrt{2q}$ ning o'rinli ekanligi kelib chiqadi. Agar $p > 2$ - tub son bo'lsa, (1.1.9) ga asosan

$$|S_2| = \sqrt{p} \quad (1.1.11)$$

o‘rinli bo‘ladi.

1.2-§. n-darajali trigonometrik yig‘indilarni baholash.

Ikkinchi tartibli ratsional trigonometrik yig‘indilarning berilgan oraliqdagi kvadratik chegirmalar va chegirma emaslar sonini aniqlashga tatbiqu.

Faraz etaylik, $p > 2$ -tub son, $Q(< P)$ -musbat butun son, R -esa $1, 2, \dots, Q$ sonlari orasidagi kvadratik chegirmalar soni, N - shu sonlar orasidagi kvadratik chegirma emaslar soni bo‘lsin. Bu yerda quyidagi teorema isbotlaymiz.

1.2.1-teorema. Ushbu formulalar o‘rinli:

$$R = \frac{Q}{2} + \frac{\theta}{2} \sqrt{p} \ln p \quad (1.2.1)$$

$$N = \frac{Q}{2} - \frac{\theta}{2} \sqrt{p} \ln p, \quad (1.2.2)$$

bu yerda $\theta \in (-1; 1)$ haqiqiy son.

Bu teoremani isbotlashda quyidagi lemmadan foydalanamiz.

1.2.1-lemma. Agar $m > 1$, M, N -butun sonlar va $M < N$ bo‘lsa, u holda

$$\Omega = \sum_{a=1}^{m-1} \left| \sum_{x=M}^N e^{\frac{2\pi i ax}{m}} \right| < m \ln m \quad (1.2.3)$$

tengsizlik o‘rinli.

Isboti. (1.1.6)-formulada $\alpha = \frac{a}{m}$, $Q+1 = M$, $Q+P = N$ deb olsak, u

holda $a \neq m$ bo‘lgani uchun

$$\left| \sum_{x=M}^N e^{\frac{2\pi i ax}{m}} \right| \leq \frac{1}{2 \left\| \frac{a}{m} \right\|}$$

hosil bo‘ladi. Agarda $1 \leq a \leq \frac{m}{2}$ bo‘lsa, u holda $\left\| \frac{a}{m} \right\| = \frac{a}{m}$ va

$\frac{m}{2} < a \leq m-1$ bo‘lsa, u holda $\left\| \frac{a}{m} \right\| = 1 - \frac{a}{m} = \frac{m-a}{m}$ bo‘ladi. Shuning uchun

ham juft m lar uchun

$$\begin{aligned} \Omega &\leq \sum_{a=1}^{m-1} \frac{1}{2 \left\| \frac{a}{m} \right\|} = \sum_{a=1}^{\frac{m}{2}} \frac{m}{2a} + \sum_{a=\frac{m}{2}+1}^{m-1} \frac{m}{2(m-a)} = \frac{m}{2} \times \\ &\times \left(\sum_{a=1}^{\frac{m}{2}} \frac{1}{a} + \sum_{a=\frac{m}{2}+1}^{m-1} \frac{1}{(m-a)} \right) = \frac{m}{2} \left(\left(\sum_{a=1}^{\frac{m-1}{2}} \frac{1}{a} + \left(\sum_{a_1=1}^{\frac{m-1}{2}} \frac{1}{a_1} + \frac{2}{m} \right) \right) \right) = m \left(\sum_{a=1}^{\frac{m}{2}} \frac{1}{a} + \frac{1}{m} \right) \end{aligned}$$

Ma’lumki, $0 < x < 1$ da

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \dots\right) > 2x.$$

Agar bunda $2x = \frac{1}{a}$ deb olsak, $\frac{1}{a} < \ln \frac{2a+1}{2a-1}$ bo‘ladi. Demak,

$$\Omega < m \left(\sum_{a=1}^{\frac{m-1}{2}} \ln \frac{2a+1}{2a-1} + \ln \frac{2m+1}{2m-1} \right).$$

Bunda

$$\sum_{a=1}^{\frac{m-1}{2}} \ln \frac{2a+1}{2a-1} + \ln \frac{2m+1}{2m-1} = \ln \frac{3}{1} + \ln \frac{5}{3} + \ln \frac{7}{5} + \dots + \ln \frac{m-1}{m-3} + \ln \frac{2m+1}{2m-1} =$$

$$\ln(m-1) + \ln \frac{2m-1+2}{2m-1} = \ln(m-1) + \ln \left(1 + \frac{2}{2m-1} \right)$$

$$\ln \left(m-1 + \frac{2(m-1)}{2m-1} \right) < \ln m$$

bo‘lganligi sababli $\Omega < m \ln m$ ni hosil qilamiz.

Endi 1.2.1-teoremani isbotlaymiz. Buning uchun

$$S = \sum_{z=1}^Q \sum_{x=0}^{p-1} \sum_{a=0}^{p-1} e^{\frac{2\pi i a(x^2-z)}{p}},$$

yig'indini qaraymiz. Agarda z kvadratik chegirma bo'lsa, $x^2 \equiv z \pmod{p}$ taqqoslama 2 ta yechimga ega. Shuning uchun ham

$$S = 2pR \quad (1.2.4)$$

Ikkinchi tomondan esa $a=0$ ga mos hadni ajratib olib S ni quyidagicha yozish mumkin:

$$S = pQ + T \quad (1.2.5)$$

bunda

$$T = \sum_{z=1}^Q \sum_{x=0}^{p-1} \sum_{a=0}^{p-1} e^{2\pi i \frac{a(x^2-z)}{p}}$$

Endi T ni baholaymiz:

$$|T| \leq \sum_{a=1}^{p-1} \left| \sum_{x=0}^{p-1} e^{2\pi i \frac{ax^2}{p}} \right| \cdot \left| \sum_{z=1}^Q e^{-2\pi i \frac{z}{p}} \right|.$$

Buning o'ng tomonida avvalo (1.1.11) tenglikdan foydalanib keyin 1.2.1-lemmani qo'llasak $|T| < p\sqrt{p} \ln p$ ga ega bo'lamiz. Bundan va (1.2.5) dan

$$S < pQ + p\sqrt{p} \ln p \quad (1.2.6)$$

tengsizlik kelib chiqadi. (1.2.6) dan (1.2.4) ga asosan

$$R < \frac{1}{2}Q + \frac{1}{2}\sqrt{p} \ln p \quad (1.2.7)$$

Tushunarliki $N = Q - R$. Shuning uchun ham (1.2.7) ga ko'ra

$$R > \frac{1}{2}Q - \frac{1}{2}\sqrt{p} \ln p \quad (1.2.8)$$

(1.2.7) va (1.2.8) lardan mos ravishda (1.2.1) va (1.2.2) baholar kelib chiqadi.

Natija. p tub moduli bo'yicha eng kichik musbat kvadratik chegirma emas $\left[\sqrt{p} \ln p \right] + 1$ dan katta emas. Bu yerda $[x]$ ifoda x ning butun qismini bildiradi.

Isboti. Haqiqatan ham, agarda $Q > \sqrt{p} \ln p$ bo'lsa, (1.2.8) dan $N > 0$ kelib chiqadi. Bu esa $2, 3, \dots, \left[\sqrt{p} \ln p \right] + 1$ sonlari orasida kamida birta p -tub moduli bo'yicha kvadratik chegirma emas mavjud degani.

1.2.2 Yetarlicha katta p lar uchun yuqoridagi natijani yaxshilash mumkin. endi biz shu bilan shug'ullanamiz.

Buning uchun bizga quyidagi lemma kerak bo'ladi.

1.2.2-lemma. Agar $n \geq 3$ va p tub son bo'lsa, ushbu

$$\eta(n) \sum_{p \leq n} \frac{\ln p}{p} < \ln n + 3,5816 + \frac{1}{2n} \ln 2\pi n + \frac{1}{12n^2},$$

tengsizlik o'rinli.

Isboti. Ma'lumki, $n!$ da p tub soni

$$\alpha_p = \left[\frac{n}{p} \right] + \left[\frac{n}{p^2} \right] + \dots + \left[\frac{n}{p^k} \right], \quad (p^k \leq n),$$

daraja bilan qatnashadi, ya'ni $n!$ ning tub ko'paytuvchilarga yoyilmasi

$$n! = \prod_{p \leq n} p^{\alpha_p},$$

dan iborat. Bu yoyilmaning ikkala tomonini logarifmlasak

$$\ln n! = \sum_{p \leq n} \alpha_p \ln p,$$

hosil bo'ladi. Bu yerda Stirling formulasiga ko'ra

$$n! = \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-\left(n+\frac{\theta}{12n}\right)}, \quad 0 < \theta < 1,$$

bo'lgani uchun

$$\ln n! = \frac{1}{2} \ln 2\pi + \left(n + \frac{1}{2} \right) \ln n - n + \frac{\theta}{12n}$$

bo'ladi. Shunday qilib,

$$n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln 2\pi + \frac{\theta}{12n} = \sum_{p \leq n} \alpha_p \ln p,$$

yoki bundan

$$n(\ln n - 1) + \frac{1}{2} \ln n + \frac{1}{2} \ln 2\pi + \frac{\theta}{12n} = \sum_{p \leq n} \alpha_p \ln p \quad (1.2.9)$$

bu yerda

$$\sum_{p \leq n} \alpha_p \ln p = \sum_{p \leq n} \left[\frac{n}{p} \right] \ln p + \sum_{p \leq n} \left(\left[\frac{n}{p^2} \right] + \dots + \left[\frac{n}{p^k} \right] \right) \ln p \quad (1.2.10)$$

(1.2.10) ning o'ng tomonidagi ikkinchi qo'shiluvchini qaraymiz. $[x] \leq x$

bo'lgani uchun

$$\begin{aligned} 0 &\leq \sum_{p \leq n} \left(\left[\frac{n}{p^2} \right] + \dots + \left[\frac{n}{p^k} \right] \right) \ln p < n \sum_{p \leq n} \left(\frac{1}{p^2} + \frac{n}{p^3} + \dots \right) \ln p = \\ &n \sum_{p \leq n} \frac{1}{p^2} \ln p = n \sum_{p \leq n} \frac{\ln p}{p^2 - p} = n \left(\sum_{p \leq 20} \frac{\ln p}{p^2 - p} + \sum_{p > 20} \frac{\ln p}{p^2 - p} \right) < \\ &n \sum_{p \leq 20} \frac{\ln p}{p(p-1)} + \sum_{p > 23} \frac{\ln k}{k(k-1)} < 0,81n + n \int_{23}^{\infty} \frac{\ln \varepsilon}{\varepsilon^4} < 0,84n \end{aligned}$$

bajariladi. Shuning uchun ham (1.2.9) va (1.2.10) lardan

$$\sum_{p \leq n} \left[\frac{n}{p} \right] \ln p = n(\ln n - 1) + \frac{1}{2} \ln 2\pi n + \frac{\theta}{12n} + 0,84\theta_1 n. \quad (1.2.11)$$

Bunda $\left[\frac{n}{p} \right] = \frac{n}{p} - \left\{ \frac{n}{p} \right\}$.

(1.2.11)- tengsizlik n butun son bo'lmasdan haqiqiy son bo'lsa ham o'z

kuchini saqlab qoladi, chunki $\left[\frac{[n]}{p} \right] = \left[\frac{n}{p} \right]$ va $[n] \ln [n] < n \ln n$.

(1.2.11) da n ni $\frac{n}{2}$ bilan almashtirib quyidagini hosil qilamiz:

$$2 \sum_{p \leq \frac{n}{2}} \left[\frac{n}{2p} \right] \ln p = 2 \frac{n}{2} \left(\ln \frac{n}{2} - 1 \right) + \ln \frac{2n\pi}{2} + \frac{\theta}{6n} + 0,84\theta_1 \frac{n}{2}.$$

Bu tengsizliklardan

$$\sum_{p \leq n} \left[\frac{n}{p} \right] \ln p - 2 \sum_{p \leq \frac{n}{2}} \left[\frac{n}{2p} \right] \ln p = n \left(\ln n - \ln \frac{n}{2} \right) - \frac{1}{2} \ln \frac{\pi n}{2} + \theta \left(\frac{1}{6n^2} + 0.42 \right) n =$$

$$n \theta \left(\left(1 + \frac{1}{2n} \right) \ln 2 + \frac{1}{6n^2} + 0.42 \right) = n \theta \left(\frac{7}{6} \ln 2 + \frac{1}{54} + 0.42 \right) = 1.2472n\theta,$$

ga ega bo‘lamiz. Bu tenglikning chap tomonining shaklini o‘zgartirib yozib olamiz. U holda

$$\sum_{\frac{n}{2} < p \leq n} \left[\frac{n}{p} \right] \ln p + \sum_{p \leq \frac{n}{2}} \left(\frac{n}{p} - \left\{ \frac{n}{p} \right\} \right) \ln p - 2 \sum_{p \leq \frac{n}{2}} \left(\frac{n}{2p} - \left\{ \frac{n}{2p} \right\} \right) \ln p = 1.2472n\theta,$$

yoki bundan

$$\sum_{\frac{n}{2} < p \leq n} \left[\frac{n}{p} \right] \ln p + \sum_{p \leq \frac{n}{2}} U_p \ln p = 1.2472n\theta.$$

Bu yerda

$$U_p = 2 \cdot \left\{ \frac{n}{2p} \right\} - \left\{ \frac{n}{p} \right\}$$

va agar $\left\{ \frac{n}{2p} \right\} < \frac{1}{2}$ bo‘lsa, $U_p = 0$ agarda $\left\{ \frac{n}{2p} \right\} \geq \frac{1}{2}$ bo‘lsa, $U_p > 0$

Shunday qilib, oxirgi tenglikning chap tomonidagi ikkinchi qo‘shiluvchi manfiy emas. U holda uni tashlab yuborib birinchi qo‘shiluvchida $\left[\frac{n}{p} \right] = 1$

bo‘lgani uchun

$$\sum_{\frac{n}{2} < p \leq n} \ln p = 1.2472\theta n,$$

ni hosil qilamiz. Bundan esa

$$\sum_{\frac{n}{2} < p \leq n} \ln p = 1.2472\theta \frac{n}{2}, \quad \sum_{\frac{n}{8} < p \leq \frac{n}{4}} \ln p = 1.2472\theta \frac{n}{4}, \dots, \quad \sum_{\frac{n}{2^k} < p \leq \frac{n}{2^{k-1}}} \ln p = 1.2472\theta \frac{n}{2^{k-1}}.$$

larni hosil qilamiz. Agar k ni $2^k > lnn$ shartni qanoatlantiruvchi qilib olsak,

$$\sum_{p \leq \frac{n}{2^k}} \ln p < \frac{n}{2^k} \ln \frac{n}{2^k} < n,$$

bo'ladi. Bulardan

$$\begin{aligned} \sum_{p \leq n} \ln p &= 1.2472\theta \left(1 + \frac{1}{2} + \dots + \frac{n}{2^{k-1}} + 1 \right) n = 1.2472\theta \left(\frac{\frac{1}{2} \left(1 - \frac{1}{2^{k-1}} \right)}{1 - \frac{1}{2}} + 2 \right) n = \\ &= 1.2472\theta \left(3 - \frac{n}{2^{k-1}} \right) n = 3.7416\theta n \end{aligned} \quad (1.2.12)$$

Endi

$$\sum_{p \leq n} \left[\frac{n}{p} \right] \ln p \leq n \sum_{p \leq n} \frac{\ln p}{p} < \sum_{p \leq n} \left(\left[\frac{n}{p} \right] + 1 \right) \ln p$$

ekanligini e'tiborga olib (1.2.11) va (1.2.12) lardan

$$n \sum_{p \leq n} \frac{\ln p}{p} = n(\ln n - 1) \frac{1}{2} \ln 2\pi n + \frac{\theta}{12n} + 0.84\theta_1 n + 3.7416\theta n$$

yoki

$$\sum_{p \leq n} \frac{\ln p}{p} = \ln n + \frac{1}{2n} \ln 2\pi n + \frac{\theta}{12n^2} + 3.5816\theta_2.$$

Shunday qilib 1.2.2-lemma isbot bo'ldi.

Endi ushbu lemmani isbotlaymiz:

1.2.3-lemma. Agar $h > 2$ va $h^\alpha \leq g < h$, $0 < \alpha < 1$ bo'lsa, u holda

$$\sum_{g < p \leq h} \frac{1}{p} = \ln \frac{\ln h}{\ln g} + 12.7248\theta \cdot \frac{1}{\ln h}.$$

Isboti. p_{k+1} bilan g dan katta eng kichik tub sonni, p_{k+l} bilan esa $p_{k+l} \leq h$ shartni qanoatlantiruvchi eng katta tub sonni belgilaylik. U holda 2.2-lemmaga asosan

$$\sum_{g < p \leq h} \frac{1}{p} = \sum_{k=1}^l \frac{\ln p_{k+l}}{p_{k+l}} \cdot \frac{1}{\ln p_{k+l}}.$$

Bu yerda

$$\frac{\ln p_{k+i}}{p_{k+i}} = \eta(p_{k+i}) - \eta(p_{k+i-1})$$

ekanligidan foydalansak va o'ng tomoniga

$$\frac{\eta(p_{k+i})}{\ln h} - \frac{\eta(p_k)}{\ln g}$$

ni qo'shib ayirsak

$$\begin{aligned} \sum_{g < p \leq h} \frac{1}{p} &= (\eta(p_{k+1}) - \eta(p_k)) \cdot \frac{1}{\ln p_{k+1}} + \\ &+ (\eta(p_{k+2}) - \eta(p_{k+1})) \cdot \frac{1}{\ln p_{k+2}} + \dots + (\eta(p_{k+l}) - \eta(p_{k+l-1})) \cdot \frac{1}{\ln p_{k+l}} = \\ &\eta(p_k) \left(\frac{1}{\ln g} - \frac{1}{\ln p_{k+1}} \right) + \eta(p_{k+1}) \left(\frac{1}{\ln p_{k+1}} - \frac{1}{\ln p_{k+2}} \right) + \dots + \eta(p_{k+l-1}) \times \\ &\times \left(\frac{1}{\ln p_{k+l-1}} - \frac{1}{\ln p_{k+l}} \right) + \eta(p_{k+l}) \left(\frac{1}{\ln p_{k+l}} - \frac{1}{\ln h} \right) + \frac{\eta(p_{k+l})}{\ln h} - \frac{\eta(p_k)}{\ln g}, \end{aligned}$$

hosil bo'ladi. Bu yerdan

$$\sum_{g < p \leq h} \frac{1}{p} = \int_g^h \frac{\eta(t) dt}{t(\ln t)^2} + \frac{\eta(h)}{\ln h} - \frac{\eta(g)}{\ln g} \quad (1.2.13)$$

bunda

$$\eta(t) = \ln t + 4.2416\theta, \quad -1 < \theta < 1, \quad t \geq 3$$

Shuning uchun ham (1.2.13) dan

$$\begin{aligned} \sum_{g < p \leq h} \frac{1}{p} &= \int_g^h \frac{\eta dt}{t \ln t} + 4.2416\theta \int_g^h \frac{\eta dt}{t(\ln t)^2} + 1 + \frac{4.2416\theta}{\ln h} - 1 - \frac{4.2416\theta}{\ln g} = \\ &= \ln \left(\frac{\ln h}{\ln g} \right) + 4.2416\theta \left(\frac{1}{\ln \frac{\ln h}{\ln g}} + \frac{1}{\ln h} + \frac{1}{\ln g} \right). \end{aligned}$$

Bu yerda $h^\alpha \leq g < h$ ekanligini, ya'ni $\alpha \ln h \leq \ln g < \ln h$ ekanligini e'tiborga olsak

$$\sum_{g < p \leq h} \frac{1}{p} = \ln \left(\frac{\ln h}{\ln g} \right) + 12.7248\theta \frac{1}{\ln h}$$

hosil bo'ladi. Shuning bilan 1.2.3-lemma isbot bo'ldi.

Endi bu isbotlangan 1.2.3-lemmani quyidagi teoremani isbotlahga tatbiq qilamiz.

1.2.2-teorema. Yetarlicha katta p –tub sonlar uchun p –tub moduli bo‘yicha eng kichik kvadratik chegirma emas $T = p^{\frac{1}{2\sqrt{e}}} (\ln p)^2$ dan katta emas.

Isboti. Faraz etaylik $1, 2, \dots, T$ sonlarinig barchasi p - tub moduli bo‘yicha kvadratik chegirma bo‘lsin. U holda har bir $Q = \sqrt{p} (\ln p)^2$ -dan katta bo‘lmagan kvadratik chegirma emas $T < q \leq Q$ shartni qanoatlantiruvchi tub bo‘luvchi q ga ega bo‘lishi kerak. Haqiqatan ham, har bir kvadratik chegirma emasning tub bo‘luvchilari orasida kamida bitta kvadratik chegirma emas mavjud bo‘lishi kerak, chunki Lejandr simboliga ko‘ra

$$\left(\frac{a_1 \cdot a_2 \cdot \dots \cdot a_k}{p} \right) = \left(\frac{a_1}{p} \right) \cdot \left(\frac{a_2}{p} \right) \cdot \dots \cdot \left(\frac{a_k}{p} \right),$$

tenglik o‘rinli bo‘lishi kerak. Shuning uchun ham Q dan katta bo‘lmagan kvadratik chegirma emaslar soni N quyidagi tengsizlikni qanoatlantiradi:

$$N \leq \sum_{T < q \leq Q} \left[\frac{Q}{q} \right] \leq Q \sum_{T < q \leq Q} \frac{1}{q}.$$

Bu munosabatning o‘ng tomonida $h = Q, g = T$ deb olib 2.3-lemmadan foydalansak

$$N \leq Q \ln \frac{\ln Q}{\ln T} + 12.7248 \cdot \frac{Q}{\ln Q} < Q \ln \frac{\frac{1}{2} \ln p + 2 \ln \ln p}{\frac{1}{2\sqrt{e}} \ln p + 2 \ln \ln p} + 12.7248 \cdot \frac{Q}{\ln Q} =$$

$$Q \left(\frac{1}{2} + \ln \frac{1 + \frac{4 \ln \ln p}{\ln p}}{1 + \frac{4\sqrt{e} \ln \ln p}{\ln p}} \right) + 12.7248 \cdot \frac{Q}{\ln Q}.$$

(1.2.14) ning o‘ng tomonida

$$\ln \frac{1 + \frac{4 \ln \ln p}{\ln p}}{1 + \frac{4\sqrt{e} \ln \ln p}{\ln p}} = \ln \left(1 + \frac{4 \ln \ln p}{\ln p} \right) - \ln \left(1 + \frac{4\sqrt{e} \ln \ln p}{\ln p} \right),$$

deb yozib olib $-1 < x < 1$ da

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + \dots$$

ekanligidan foydalanamiz. U holda yetarlicha katta p – tub sonlar uchun

$$\begin{aligned} \ln \left(1 + \frac{4 \ln \ln p}{\ln p} \right) - \ln \left(1 + \frac{4\sqrt{e} \ln \ln p}{\ln p} \right) &= \left(\frac{4 \ln \ln p}{\ln p} - \frac{1}{2} \left(\frac{4 \ln \ln p}{\ln p} \right)^2 + \dots \right) - \\ &\left(\frac{4\sqrt{e} \ln \ln p}{\ln p} - \frac{1}{2} \left(\frac{4\sqrt{e} \ln \ln p}{\ln p} \right)^2 + \dots \right) = (4 - 4\sqrt{e}) \frac{\ln \ln p}{\ln p} + \frac{16e - 16}{2} \left(\frac{\ln \ln p}{\ln p} \right)^2 = \\ &(4 - 4\sqrt{e}) \frac{\ln \ln p}{\ln p} + \frac{16e - 16}{2} \left(\frac{\ln \ln p}{\ln p} \right)^2 + \\ &= -(4\sqrt{e} - 4) \frac{\ln \ln p}{\ln p} + \theta \frac{0.001}{\ln p} \end{aligned}$$

bajariladi. (1.2.14) da bundan va $Q = \sqrt{p}(\ln p)^2$ ekanligidan foydalansak

$$N < Q \left(\frac{1}{2} - \frac{(4\sqrt{e} - 4) \ln \ln p}{\ln p} \right) + 25.45 \frac{Q}{\ln p},$$

hosil bo‘ladi. Ikkinchi tomondan 1.2.1-teoreмага asosan

$$N > \frac{Q}{2} - \frac{1}{2} \sqrt{p} \ln p.$$

Demak,

$$\frac{Q}{2} - \frac{Q}{2 \ln p} < Q \left(\frac{1}{2} - \frac{(4\sqrt{e} - 4) \ln \ln p}{\ln p} \right) + 25.45 \frac{Q}{\ln p}$$

Bundan $(4\sqrt{e} - 4) \ln \ln p$ ifoda chegaralangan bo‘lishi kerak degan xulosa kelib chiqadi. Bunday bo‘lishi mumkin emas. Bu qarama-qarshilik teskari farazimizdan kelib chiqdi. Demak, 1.2.2-teorema o‘rinli.

1.3-§. Trigonometrik yig'indilarning berilgan oraliqdagi n-darajali chegirmalar sonini aniqlashga tatbiqi.

Sonlar nazariyasida g - eng kichik musbat boshlang'ich ildizni yuqoridan baholash muhim ahamiyatga ega. I. M. Vinogradov va B. I. Segallarning ishlarida bunday baho mavjud. Ular yetarlicha katta p tub soni va ixtiyoriy $\varepsilon > 0$ son uchun $g = O(p^{\frac{1}{2}+\varepsilon})$ ekanligini isbotlaganlar. g uchun olingan bahoda "O" -simvoli qatnashganligi uchun undan aniq sonli hisoblashlarda foydalanib bo'lmaydi. Ushbu ishda g uchun "O"-simvoliga bog'liq bo'lmagan aniqlashtirilgan $g \leq (4^k \sqrt{p} \ln p) + 1$ baho isbotlaymiz. Bu yerda k soni $p-1$ ning tub bo'luvchilari soni.

Faraz etaylik, $p \geq 3$ tub son bo'lsin. L bilan p moduli bo'yicha $1, 2, 3, \dots, Q$ sonlari orasida joylashgan boshlang'ich ildizlar sonini belgilaymiz. Bu yerda $Q < p$. g shu boshlang'ich ildizlardan bittasi bo'lsin. U holda $a \equiv g^\gamma \pmod{p}$ boshlang'ich ildiz bo'lishi uchun $(\gamma, p-1) = 1$ bo'lishi zarur va yetarlidir.

Faraz etaylik a soni $1, 2, 3, \dots, Q$ qiymatlarni qabul qilsin. Har qanday a ga $\gamma = \text{inda}$ va $\delta = (\gamma, p-1)$ soni mos keladi. U holda osonlik bilan tekshirib ko'rish mumkinki,

$$L = \sum_{a=1}^Q \sum_{dd_1=\delta} \mu(d) = \sum_{dd_1=p-1} \mu(d) S_d, \quad (1.3.1)$$

tenglik o'rinli. Bu yerda

$$\mu(d) = \begin{cases} 1, & \text{agar } d = 1 \text{ bo'lsa,} \\ (-1)^l, & \text{agar } d = p_1 p_2 \dots p_l \text{ bo'lsa,} \\ 0, & \text{agarda } p^2 \nmid d \text{ bo'lsa} \end{cases}$$

tenglik bilan aniqlanuvchi Myobus funksiyasi, S_d esa δ ning d ga karrali

bo'ladigan a lar sonini bildiradi. δ ning d ga karrali bo'lishi uchun γ ning d ga karrali bo'lishi zarur va yetarlidir.

Shunday qilib, S_d bu $1, 2, 3, \dots, Q$ sonlari orasidagi d -darajali chegirmalar soniga teng. Shuning uchun ham

$$S_d = \frac{Q}{d} + \theta \sqrt{p} \ln p$$

bo'ladi. Buni (1.3.1) qo'yib

$$L = Q \sum_{d|p-1} \frac{\mu(d)}{d} + \theta \sigma \sqrt{p} \ln p,$$

ni hosil qilamiz. Bu yerda σ bilan $p-1$ ning birdan katta sonning kvadratiga bo'linmaydigan bo'luvchilari soni belgilangan. U holda quyidagi tenglik o'rinli:

$$L = \frac{\varphi(p-1)}{p-1} Q + \theta \sigma \sqrt{p} \ln p. \quad (1.3.2)$$

Bu yerda multiplikativ funksiyalarning asosiy ayniyatiga ko'ra $\theta(n)$ multiplikativ funksiya va $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ bo'lsa,

$$\sum_{d|n} \theta(d) = \prod_{i=1}^k (1 + \theta(p_i) + \theta(p_i^2) + \dots + \theta(p_i^{\alpha_i})), \quad (1.3.3)$$

tenglik o'rinli, (1.3.3) da $n = p-1$, $\theta(d) = \frac{\mu(d)}{d}$ deb olib, agar $\alpha_i > 1$

bo'lsa $\mu(p^{\alpha_i}) = 0$ va $\alpha_i = 1$ bo'lganda $\mu(p_i) = -1$ ekanligidan foydalansak

$$\sum_{d|(p-1)} \frac{\mu(d)}{d} = \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right),$$

kelib chiqadi.

$$\varphi(p-1) = (p-1) \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right),$$

bo'lganidan

$$\sum_{d|(p-1)} \frac{\mu(d)}{d} = \frac{\varphi(p-1)}{p-1}, \quad (1.3.4)$$

bo‘ladi. (1.3.1) dan (1.3.2) ni keltirib chiqarishda biz (1.3.4) ni inobatga oldik.

Endi eng kichik musbat boshlang‘ich ildizning tartibini aniqlash uchun bizga, k ning bo‘luvchilari soni $\tau(k)$ ning bahosi kerak bo‘ladi.

$k = p^\alpha$ (bu yerda p tub son, $\alpha \geq 1$ – butun son) va ixtiyoriy $\varepsilon > 0$ soni uchun

$$\frac{\tau(k)}{k^\varepsilon} = \frac{\alpha + 1}{k^\varepsilon} \leq \frac{2\alpha}{k^\varepsilon} = \frac{2 \ln p^\alpha}{k^\varepsilon \ln p} \leq \frac{2 \ln k}{\ln 2 k^\varepsilon},$$

deb yoza olamiz. Shuning uchun ham, agar $k = p^\alpha$ bo‘lsa,

$$\lim_{k \rightarrow \infty} \frac{\tau(k)}{k^\varepsilon} = 0, \quad (1.3.5)$$

bajariladi.

Endi umumiy holni qaraymiz. $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ bo‘lsin. $\varepsilon > 0$ doimiy sonni tanlab olib, $\frac{\tau(k)}{k^\varepsilon}$ nisbatni qaraymiz. $\tau(k)$ multiplikativ funksiya bo‘lgani uchun

$$\frac{\tau(k)}{k^\varepsilon} = \frac{\tau(p_1^{\alpha_1})}{p_1^{\alpha_1 \varepsilon}} \frac{\tau(p_2^{\alpha_2})}{p_2^{\alpha_2 \varepsilon}} \dots \frac{\tau(p_l^{\alpha_l})}{p_l^{\alpha_l \varepsilon}}, \quad (1.3.6)$$

bajariladi. (1.3.5) dan quyidagilar kelib chiqadi:

- a). (1.3.6) ning o‘ng tomonidagi har bir ko‘paytuvchi chegaralangan;
- b). har qanday $\delta > 0$ soni uchun shunday $N(\delta)$ topiladiki, $p_i^{\alpha_i} > N(\delta)$ bo‘lganda (1.3.6) o‘ng tomonidagi unga mos ko‘paytuvchi δ dan kichik bo‘ladi;
- c). xususiyl holda $\delta = 1$ bo‘lganda, shunday N_1 doimiy soni mavjud bo‘lib $p_i^{\alpha_i} > N_1$ bo‘lganda (1.3.6) ning o‘ng tomonidagi unga mos keluvchi ko‘paytuvchi 1 dan kichik bo‘ladi.

N_1 dan oshmaydigan $p_1^{\alpha_1}, \dots, p_l^{\alpha_l}$ lar soni chegaralangan bo‘lganligi sababli a) va b) larga ko‘ra (1.3.6) ning o‘ng tomonidagi ixtiyoriy sondagi ko‘paytuvchilarning ko‘paytmasi biror c doimiy sonidan kichik bo‘ladi.

Ikkinchi tomondan esa, yetarlicha katta $k > N(\delta)$ lar uchun hech bo‘lmasa $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ lardan birortasi $N\left(\frac{\delta}{c}\right)$ dan katta bo‘ladi. b) ga ko‘ra (1.3.6) ning

o'ng tomonidagi bunga mos bo'lgan ko'paytuvchi $\frac{\delta}{c}$ dan kichik bo'ladi va (1.3.6)

ning o'ng tomoni δ dan kichik bo'ladi. Bulardan

$$\lim_{k \rightarrow \infty} \frac{\tau(k)}{k^\varepsilon} = 0,$$

kelib chiqadi. Bundan keyin biz $\tau(k) \leq c_1(\varepsilon) \cdot k^\varepsilon$ bahodan foydalanamiz. Bu yerda

$c_1(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2^\varepsilon}}$ deb olish mumkin. Lekin [12] da $c_1(\varepsilon)$ uchun yanada aniqroq qiymat

$$c_1(\varepsilon) = \prod_{p_i < 2^{\frac{1}{\varepsilon}}} \max \left\{ \left[\frac{1}{\varepsilon \ln p_i} \right] p_i^{-\varepsilon \left[\frac{1}{\varepsilon \ln p_i} - 1 \right]}; \left[\frac{1}{\varepsilon \ln p_i} + 1 \right] p_i^{-\varepsilon \left[\frac{1}{\varepsilon \ln p_i} \right]} \right\}, \quad (1.3.7)$$

olingan. $\tau(k)$ uchun topilgan bahodan foydalanib, Eyler funksiyasi $\varphi(k)$ uchun ham baho olish mumkin.

Tushunarliki,

$$\varphi(k) = k \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_l}\right) \geq k \frac{1}{2^l}$$

Lekin, 2^l bu $p_1 p_2 \dots p_l$ sonining bo'luvchilar soni bo'lgani uchun $2^l \leq \tau(k) \leq c_1(\varepsilon) k^\varepsilon$. Shuning uchun ham $\varphi(k) \geq \frac{k}{2^l} \geq \frac{1}{c_1(\varepsilon)} k^{1-\varepsilon}$. Bu yerda $\varepsilon > 0$

ixtiyoriy fiksirlangan son bo'lib c esa faqat ε ga bog'liq doimiy.

Endi

$$\sigma \leq \tau(p-1) < c_1 \left(\frac{\varepsilon}{4}\right) (p-1)^{\frac{\varepsilon}{4}}$$

va

$$\frac{\varphi(p-1)}{p-1} \geq \frac{1}{c_1\left(\frac{\varepsilon}{4}\right)} \frac{(p-1)^{1-\frac{\varepsilon}{4}}}{p-1} = \frac{1}{c_1\left(\frac{\varepsilon}{4}\right)} (p-1)^{-\frac{\varepsilon}{4}},$$

ekanligini e'tiborga olib, (1.3.2) formuladan

$$L \geq \frac{Q}{c_1\left(\frac{\varepsilon}{4}\right)} (p-1)^{-\frac{\varepsilon}{4}} - c_1\left(\frac{\varepsilon}{4}\right) (p-1)^{\frac{\varepsilon}{4}} \sqrt{p} \ln p$$

tengsizlikka ega bo'lamiz.

Agar bu oxirgi tengsizlikda

$$Q = c_1^2\left(\frac{\varepsilon}{4}\right) (p-1)^{\frac{\varepsilon}{2}} \sqrt{p} \ln p$$

deb olsak $L > 0$ bo'ladi. Bu degan so'z $1, 2, 3, \dots, Q$ sonlari orasida hech bo'lmasa bitta $p \geq 3$ modul bo'yicha boshlag'ich ildiz mavjud degani va demak, $p \geq 3$ moduli bo'yicha eng kichik musbat boshlang'ich ildiz

$$g \leq c_1^2\left(\frac{\varepsilon}{4}\right) (p-1)^{\frac{\varepsilon}{4}} \sqrt{p} \ln p, \quad (1.3.8)$$

shartni qanoatlantiradi.

I. M. Vinogradov va B. I. Segallarning ishlarida esa yetarlicha katta p va ixtiyoriy $\varepsilon > 0$ son uchun

$$g = O\left(p^{\frac{1}{2}+\varepsilon}\right), \quad (1.3.9)$$

ekanligi isbotlangan edi. (1.3.8) va (1.3.9) baholar tartib jihatidan bir xil bo'lsada (1.3.8) da berilgan p va ixtiyoriy $\varepsilon > 0$ son uchun $c_1\left(\frac{\varepsilon}{4}\right)$ ni aniq hisoblab g uchun aniq chegara ko'rsata olamiz, lekin (1.3.9) da bu ishni qilib bo'lmaydi. Shuningdek (1.3.8)-tengsizlik barcha $p \geq 3$ tub sonlar uchun o'rinli. (1.3.2) formulada va σ ning ta'rifidan foydalanib $p \geq 3$ modul bo'yicha eng kichik musbat boshlang'ich ildiz uchun yuqoridan baho olish mumkin. (1.3.2) da σ bu $(p-1)$ ning tub bo'luvchilari soni, shuning uchun ham, agar $(p-1)$ ning

kanonik yoyilmasi

$$p-1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

bo'lsa, u holda

$$\tau(p-1) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1)$$

bo'lib $\sigma = 2^k$ bo'ladi.

Shuningdek,

$$\varphi(p-1) = (p-1) \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right) > (p-1) \frac{1}{2^k}$$

ekanligini e'tiborga olib (2) dan

$$L > 2^{-k} Q - 2^k \sqrt{p} \ln p, \quad (1.3.10)$$

ga ega bo'lamiz.

Endi, agar $Q = (4^k \sqrt{p} \ln p) + 1$ deb olsak, $L > 0$ bo'ladi va demak bundan (1.3.8) dagi singari

$$g \leq (4^k \sqrt{p} \ln p) + 1 \quad (1.3.11)$$

degan xulosaga kelamiz, (1.3.11)-formula sodda bo'lsada undan foydalanish uchun $(p-1)$ ning har xil tub bo'luvchilari sonini bilish kerak. Agar $(p-1)$ ning tub bo'luvchilari soni aniq bo'lmasa, u holda (1.3.8) dan foydalanish ma'qulroqdir. (1.3.8) dan foydalanganda $c_1 \left(\frac{\varepsilon}{4}\right)$ ning qiymatini (1.3.7) tenglikdan foydalanib hisoblash yaxshi natijaga olib keladi.

I bob uchun xulosa

I bobda birinchi darajali va ikkinchi darajali trigonometrik yig'indilarni atrioflichga o'rganildi va baholar olindi.

Ushbu

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}}$$

yig'indini qaraymiz. Bunda a – butun son, m -natural son. S_1 yig'indidagi $e^{2\pi i \xi} = \cos 2\pi \xi + i \sin 2\pi \xi$ ko'rinishdagi funksiya davriy bo'lib uning davri 1 ga teng, ya'ni $e^{2\pi i(\xi+1)} = e^{2\pi i \xi} \cdot e^{2\pi i} = e^{2\pi i \xi}$

Bundan agar $x \equiv y \pmod{m}$ ya'ni $x = y + mt, t \in \mathbb{Z}$ bo'lsa, u holda

$$e^{2\pi i \frac{ax}{m}} = e^{2\pi i \frac{a(y+mt)}{m}} = e^{2\pi i \frac{ay}{m}} \cdot e^{2\pi i at} = e^{2\pi i \frac{ay}{m}}$$

ekanligi kelib chiqadi. Agarda S_1 yig'indida a soni m ga bo'linsa (a/m), u holda $a = mt$ va

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i tx} = 1 + e^{2\pi i t} + e^{2\pi i (2t)} + \dots + e^{2\pi i (m-1)t} = \underbrace{1 + 1 + \dots + 1}_m = m,$$

bo'ladi. Agarda S_1 yig'indida a soni m ga bo'linmasa ($a \nmid m$), u holda

$$S_1 = 1 + e^{2\pi i \frac{a}{m}} + \dots + e^{2\pi i \frac{a(m-1)}{m}} = \frac{1 - \left(e^{2\pi i \frac{a}{m}} \right)^m}{1 - e^{2\pi i \frac{a}{m}}} = \frac{1 - e^{2\pi i a}}{1 - e^{2\pi i \frac{a}{m}}} = 0.$$

Shunday qilib,

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}} = \begin{cases} m, & \text{agar } a/m, \\ 0, & \text{agar } a \nmid m \end{cases}$$

ekanligini isbotladik.

$$S_2 = \sum_{x=0}^{q-1} e^{\frac{2\pi i a}{q} x^2}, (a, q) = 1 \text{ ushbu ikkinchi darajali yig'indi uchun quyidagi}$$

bahoni oldik (1.1.9) va (1.1.10) dan hamma vaqt $|S_2| \leq \sqrt{2q}$ ning o'rinli ekanligi kelib chiqadi

**II-BOB. NATURAL SONLAR BO‘YICHA OLINGAN
TRIGONOMETRIK YIG‘INDILARNING ADDITIV MASALALARNI
YECHISHGA TATBIQLARI**

2.1-§. Veyl teoremasi va uning tatbiqlari

Veyl yig‘indilarini ko‘rib chiqishga o‘tamiz

$$S_1 = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)}, \quad f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \dots + \alpha_{n-1} x + \alpha_n \quad . \quad (2.1.1)$$

Bu yerda Q - butun son, P - musbat butun son, $\alpha, \alpha_1, \dots, \alpha_{n-1}, \alpha_n$ -ixtiyoriy haqiqiy sonlar, $n \geq 2$. Ushbu yig‘indi uchun biz birinchi navbatda bitta tengsizlikni olamiz, uning yordamida biz yetakchi α koeffitsiyentining ratsional kasrga yaqinlashish xususiyatiga qarab yakuniy bahoni topamiz.

Bu tengsizlikni chiqarishda biz $F(x)$ funksiyaning chekli ayirmasi tushunchasidan foydalanamiz. Ifodalar

$$\Delta_{y_1} F(x) = F(x + y_1) - F(x)$$

$$\Delta_{y_2} \Delta_{y_1} F(x) = \Delta_{y_1} F(x + y_2) - \Delta_{y_1} F(x)$$

$$\Delta_{y_3} \Delta_{y_2} \Delta_{y_1} F(x) = \Delta_{y_2} \Delta_{y_1} F(x + y_3) - \Delta_{y_2} \Delta_{y_1} F(x)$$

.....

Bu yerda y_1, y_2, y_3, \dots - ixtiyoriy sonlar, mos ravishda, $F(x)$ funksiyaning birinchi, ikkinchi, uchinchi va hokazo tartibining chekli ayirmasi deyiladi. Ko‘pincha k - tartibidagi chekli farq oddiygina k – ayirma deb ataladi. $f(x)$ yuqorida aniqlangan ko‘phad, degan faraz ostidagi birinchi $f(x)$ ayirmani topamiz.

Bu holatdan

$$\Delta f(x) = \alpha(x + y_1)^n + \alpha_1(x + y_1)^{n-1} + \dots + \alpha_n - (\alpha_0 x^n + \alpha_1 x^{n-1} + \dots + \alpha_{n-1} x + \alpha_n) =$$

$$\begin{aligned}
& +(\alpha_{n-1}(x + y_1) - \alpha_{n-1}x = \\
& = ny_1\alpha x^{n-1} + \frac{n(n-1)}{1 \cdot 2} y_1^2 \alpha x^{n-2} + \dots + (n-1)y_1\alpha_1 x^{n-2} + \dots \\
& = ny_1\alpha x^{n-1} + y_1 \left(\frac{n(n-1)}{1 \cdot 2} y_1 \alpha + (n-1)\alpha_1 \right) x^{n-2} + \dots \\
& = (\alpha(x + y_1)^n - \alpha x^n) + (\alpha_1(x + y_1)^{n-1} - \alpha_1 x^{n-1}) + \dots + \alpha_n - (\alpha_0 x^n + \alpha_1 x^{n-1} + \dots + \\
& \qquad \qquad \qquad (2.1.2)
\end{aligned}$$

Demak, n -darajali ko'phadning yetakchi koeffitsiyenti α bilan birinchi ayirmasi, yetakchi koeffitsiyenti $\alpha n y_1$ bo'lgan $(n-1)$ -darajali ko'phaddir. Bundan foydalanib, n darajali ko'phadning yetakchi koeffitsiyenti α k -ayirmasi, $n-k$ darajali ko'phad yetakchi koeffitsiyenti $\alpha n y_1$ ekan. $k = n-1$ da yetakchi koeffitsiyent α bilan biz $(n-1)$ -ayirmasi x koeffitsiyenti $\alpha n! y_1 y_2 \dots y_{n-1}$ ga teng bo'lgan x ning chiziqli funksiyasi ekanligini topamiz.

(2.1.2) da ko'rishimiz mumkinki, birinchi ayirmaning barcha koeffitsiyentlari y_1 ko'paytuvchini o'z ichiga olganligi kelib chiqadi. Shuning uchun k -ayirma koeffitsiyentlarining har birida ko'paytuvchi sifatida $y_1, y_2, y_3, \dots, y_k$ -ko'paytmasi borligini xulosa qilish oson.

Quyida biz Koshi tengsizligidan foydalanamiz. $\xi_1, \xi_2, \xi_3, \dots, \xi_m$ ixtiyoriy haqiqiy sonlar bo'lsin. Bundan

$$(\xi_1 + \xi_2 + \xi_3 + \dots + \xi_m)^2 \leq m(\xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_m^2).$$

Quyidagi tengsizlikning o'rinaliligi kvadrat tenglamaning ikkita haqiqiy ildizga ega bo'lishi mumkin emasligidan kelib chiqadi:

$$\begin{aligned}
& (z + \xi_1)^2 + (z + \xi_2)^2 + \dots + (z + \xi_m)^2 = \\
& = mz^2 + 2(\xi_1 + \xi_2 + \xi_3 + \dots + \xi_m)z + \xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_m^2 = 0.
\end{aligned}$$

Shuning uchun uning diskriminanti

$$4(\xi_1 + \xi_2 + \xi_3 + \dots + \xi_m)^2 - 4m(\xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_m^2),$$

musbat bo'lishi mumkin emas.

Veyl yig'indilari uchun ko'rsatilgan tengsizlikni chiqarishga o'tamiz. Bizda quyidagi amal bor

$$|S|^2 = S\bar{S} = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x_1)} = \sum_{x_1=Q+1}^{Q+P} \sum_{x_2=Q+1}^{Q+P} e^{2\pi i (f(x_1) - f(x_2))}.$$

Keling, almashtirish kiritamiz

$$x_1 = x_3 + y_1.$$

x_1 va x_2 ni qo'shganda $Q+1, Q+2, \dots, Q+P$ qiymatlari bir-biridan mustaqil ishlaydi, keyin $x_2 + y_1$ va x_2 quyidagi shartlarni qanoatlantiradigan butun son qiymatlarni qabul qiladi.

$$Q+1 \leq x_2 + y_1 \leq Q+P, \quad Q+1 \leq x_2 \leq Q+P.$$

Shunday qilib, y_1 quyidagi shartni qanoatlantiradigan barcha butun qiymatlarni qabul qiladi

$$-P+1 \leq y_1 \leq P-1,$$

va bu qiymatlarning har biri uchun

$$y_1 \geq 0 \text{ bo'lganda } Q+1 \leq x_2 \leq Q+P - y_1$$

$$y_1 < 0 \text{ bo'lganda } Q+1 - y_1 \leq x_2 \leq Q+P$$

shartni qanoatlantiradigan barcha butun son qiymatlari oladi. Shunday qilib, y_1 ning har bir qiymati uchun x_2 butun sonlarning P tadan ko'p bolmagan ketma-ket qiymatlarini oladi, deb aytishimiz mumkin.

$$|S|^2 = \sum_{y_1=-P+1}^{P-1} \sum_{x_2=Q+1}^{Q+P} e^{2\pi i \Delta f(x_2)} \leq \sum_{y_1=-P+1}^{P-1} \left| \sum_{x_2=Q+1}^{Q+P} e^{2\pi i \Delta f(x_2)} \right|.$$

Bu yerda Q_1 va P_1 y_1 ga bog'liq va har bir P_1 musbat va P dan oshmaydi.

Koshi tengsizligini oxirgi munosabatga qo'llasak

$$|S|^4 \leq (2P-1) \sum_{y_1=-P+1}^{P-1} \left| \sum_{x_2=Q+1}^{Q+P} e^{2\pi i \Delta f(x_2)} \right|^2 = (2P-1) \sum_{y_1=-P+1}^{P-1} \left| \sum_{x_3=Q+1}^{Q+P} e^{2\pi i (\Delta f(x_3) - \Delta f(x_2))} \right|^2$$

$x_2 = x_3 + y_2$ almashtirish kiritib va yuqoridagidek fikr yuritib topamiz

$$|S|^4 \leq (2P-1) \sum_{y_1=-P+1}^{P-1} \sum_{x_{k+1}=Q_k+1}^P \sum_{x_k=Q_k+1}^{Q_k+P_k} e^{2\pi i \Delta \dots \Delta f(x_{k+1})} = (2P-1) \sum_{y_1=-P+1}^{P-1} \sum_{y_2=-P_1+1}^{P_1-1} \sum_{x_3=Q_2+1}^{Q_2+P_2} e^{2\pi i \Delta \Delta f(x_3)} \leq$$

$$(2P-1) \sum_{y_1=-P+1}^{P-1} \sum_{y_2=-P_1+1}^{P_1-1} \left| \sum_{x_3=Q_2+1}^{Q_2+P_2} e^{2\pi i \Delta \Delta f(x_3)} \right|^2.$$

Bu yerda Q_2 va P_2, y_1 va y_2 ga bog‘liq va har bir P_2 musbat, P dan oshmaydi. Ushbu jarayonni ($k \leq n-1$)-marta qo‘llagandan so‘ng quyidagini topamiz:

$$\begin{aligned} |S|^{2^k} &\leq (2P-1)^{2^k-k-1} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_k=-P_{k-1}-1}^{P_{k-1}-1} \sum_{x_{k+1}=Q_k+1}^{Q_k+P_k} e^{\frac{2\pi i \Delta \dots \Delta f(x_{k+1})}{y_k y_1}} \leq \\ &\leq (2P-1)^{2^k-k-1} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_k=-P_{k-1}-1}^{P_{k-1}-1} \left| \sum_{x_{k+1}=Q_k+1}^{Q_k+P_k} e^{\frac{2\pi i \Delta \dots \Delta f(x_{k+1})}{y_k y_1}} \right|. \end{aligned}$$

Bu yerda Q_k va P_k lar y_1, y_2, \dots, y_k ga bog‘liq va har bir P_k musbat va P dan oshmaydi. Xususan, $k = n-1$ uchun olamiz

$$|S|^{2^{n-1}} \leq (2P-1)^{2^{n-1}-n} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_{n-1}=-P_{n-2}+1}^{P_{n-2}-1} \left| \sum_{x_n=-Q_{n-1}+1}^{Q_{n-1}+P_{n-1}} e^{\frac{2\pi i \Delta \dots \Delta f(x_n)}{y_{n-1} y_1}} \right|.$$

Lekin $\frac{\Delta \dots \Delta f(x_n)}{y_{n-1} y_1}$ birinchi darajali ko‘phad bo‘lgani uchun x_n yetakchi koeffitsiyenti $\alpha n! y_1 \dots y_{n-1}$ bilan, keyin

$$|S|^{2^{n-1}} \leq (2P-1)^{2^{n-1}-n} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_{n-1}=-P_{n-2}+1}^{P_{n-2}-1} \left| \sum_{x_n=-Q_{n-1}+1}^{Q_{n-1}+P_{n-1}} e^{\frac{2\pi i \Delta \dots \Delta f(x_n)}{y_{n-1} y_1}} \right|.$$

Yoki

$$\begin{aligned} |S|^{2^{n-1}} &\leq (2P-1)^{2^{n-1}-n} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_{n-1}=-P_{n-2}+1}^{P_{n-2}-1} \min \left(P_{n-1}, \frac{1}{2(\alpha n! y_1 \dots y_{n-1})} \right). \\ &\leq (2P-1)^{2^{n-1}-n} \sum_{y_1=-P+1}^{P-1} \dots \sum_{y_{n-1}=-P_{n-2}+1}^{P_{n-2}-1} \min \left(P, \frac{1}{2(\alpha n! y_1 \dots y_{n-1})} \right). \end{aligned}$$

$y_1 \dots y_{n-1}$ ko‘paytma nolga aylanadigan o‘ng tomondagi qo‘shiluvchilar soni quyidagi ifodadan oshmasligini tekshirish oson.

$$\begin{aligned} 1 + (n-1)(2P-1) + \frac{(n-1)(n-2)}{1*2} (2P-1)^2 + \dots + \\ + (n-1)(2P-1)^{n-2} < 2^{n-1} (2P-1)^{n-2}. \end{aligned}$$

Biz har bir bunday hadlarni P orqali almashtiramiz. Qolganlar qo‘shiluvchilar har biriga yig‘indilar $y_1 \dots y_{n-1}$ ko‘paytma ishorasiga bog‘liq emas

$$|S|^{2^{n-1}} \leq (2P-1)^{2^{n-1}-n} \cdot 2^{n-1} (2P-1)^{n-2} \cdot P +$$

$$\begin{aligned}
& +(2P-1)^{2^{n-1}-n} \cdot 2^{n-1} \sum_{y_1=1}^{P-1} \dots \sum_{y_{n-1}=1}^{P-1} \min\left(P, \frac{1}{2(\alpha n! y_1 \dots y_{n-1})}\right) < \\
& < 2^{2^{n-1}+n-3} \left(\left(P^{2^{n-1}-1} + P^{2^{n-1}-n} \sum_{y_1=1}^P \dots \sum_{y_{n-1}=1}^P \min\left(P, \frac{1}{2(\alpha n! y_1 \dots y_{n-1})}\right) < \right) \right).
\end{aligned}$$

Bu ifodaga e'tibor qaratsak

$$2^{2^{n-1}+n-3} < 3^{2^{n-1}}.$$

Kerakli tengsizlikni topamiz:

$$|S|^{2^{n-1}} \leq 3^{2^{n-1}} \left(P^{2^{n-1}-1} + P^{2^{n-1}-n} \sum_{y_1=1}^P \dots \sum_{y_{n-1}=1}^P \min\left(P, \frac{1}{2(\alpha n! y_1 \dots y_{n-1})}\right) < \right) \quad (2.1.3)$$

Veyl yig'indilarning yakuniy bahosini topishga o'tishdan oldin,

$$y_1 y_2 y_3 \dots y_k = M \quad (2.1.4)$$

Bu yerda $y_1, y_2, y_3, \dots, y_k$ lar butun son, $k \geq 2$.

Noma'lum tenglamaning yechimlar soni T uchun M ga bog'liq quyidagi ifoda o'rinli boladi

$$T = O(M^\varepsilon),$$

bu yerda ε ixtiyoriy kichik musbat doimiy. Darhaqiqat, (2.1.4) tenglamani qanoatlantiruvchi y_1, y_2, \dots, y_k sonlarining har biri M sonining bo'luvchisidir. Shuning uchun bu sonlarni har birining mumkin bo'lgan qiymatlari soni quyidagicha bo'ladi;

$$\tau(M) = O(M^{\frac{\varepsilon}{k}}).$$

Shuning uchun (2.1.4) tenglamaning yechimlari soni

$$T = O(M^{\frac{\varepsilon-k}{k}}) = O(M^\varepsilon),$$

ifoda bilan aniqlanadi.

Quyida biz n ni doimiy son deb olamiz. Bundan tashqari, biz ba'zan $A = O(B)$ o'rnida $A \ll B$ yozamiz.

(2.1.3) ifodadagi $n!y_1\dots y_{n-1}$ natija $n!P^{n-1}$ dan oshmaydigan butun qiymatlarni qabul qiladi va ko‘rib chiqilayotgan ifodaning har bir qiymati quyidagi natijadan ortiq bo‘lmaydi

$$T = O(P^{\varepsilon_1})$$

Bu yerda ε_1 musbat doimiy. Shuning uchun,

$$|S|^{2^{n-1}} \leq P^{2^{n-1}-1} + P^{2^{n-1}-n-\varepsilon_1} \sum_{y=1}^{n!P^{n-1}} \min\left(P, \frac{1}{(\alpha, y)}\right) \quad (2.1.5)$$

Ratsional kasr yordamida α sonining yaqinlashuvi quyidagi ifoda bilan aniqlanadi.

$$\alpha = \frac{a}{q} + \frac{1}{q^2}, \quad q > 0, (a, q) = 1, |\lambda| \leq \Lambda, \Lambda \geq 1 \quad (2.1.6)$$

Ma’lumki, har qanday haqiqiy son uchun $\frac{a}{q}$ shartni qanoatlantiradigan cheksiz sonli kasrni topish mumkin, $\Lambda = 1$ bo‘lsa ham. Biz (2.1.5) ifodaning o‘ng tomonini qisman yig‘indilarga ajratamiz, shunda har bir bunday yig‘indida (oxirgisidan tashqari) y ketma-ket q qiymatni oladi, va oxirgi ifodada y ketma-ket q dan kam qiymatlarni qabul qilishi mumkin. Ushbu orqali qisman yig‘indilar soni quyidagi ifodadan oshmaydi,

$$\left[\frac{n!P^{n-1}}{q} \right] + 1 = O\left(\frac{P^{n-1}}{q} + 1 \right) \quad (2.1.7)$$

va har biri quyidagi ko‘rinishda keladi:

$$L = \sum_{y=R+1}^{R+q} \min\left(P, \frac{1}{(\alpha, y)}\right)$$

Bu yerda R ifoda $0, q, 2q, \dots$, qiymatlarni oladi va biz qisman yig‘indilarni bir xil shaklda deb taxmin qilishimiz mumkin. Chunki shartlar sonining ko‘payishi bilan yig‘indi faqat ortishi mumkin. $y = R + s$ o‘zgartirish kiritish orqali, quyidagiga ega bo‘lamiz

$$L = \sum_{s=1}^q \min\left(P, \frac{1}{(R\alpha + \alpha s)}\right)$$

Ammo $R\alpha$ soni qanday bo‘lishidan qat’iy nazar

$$R\alpha = \frac{b}{q} + \frac{\theta}{q}, \quad |\theta| < 1$$

ifoda qanoatlantiradigan b butun sonni tanlashimiz mumkin. λ songa qarama qarshi θ son bor deb tahmin qilish mumkin. Oxirgi ifodadan va (2.1.6) dan

$$\begin{aligned} (R\alpha + as) &= \left(\frac{b}{q} + \frac{\theta}{q} + \frac{as}{q} + \frac{\lambda s}{q^2} \right) = \\ &= \left(\frac{as + b + \frac{\lambda s}{q} + \theta}{q} \right) = \left(\frac{v(s) + \theta_s \Lambda}{q} \right) \end{aligned} \quad (2.1.8)$$

Bu yerda $v(s)$ q modul bo‘yicha $as + b$ ning eng kichik musbat chegirmasi va $|\theta_s| < 1$. $(a, q) = 1$ bolgani uchun $as + b$ S bilan q modul bo‘yicha to‘liq chegirmalar sinfini tashkil qiladi. Shuning uchun $v(s)$ har bir yig‘indida L ifoda $0, 1, \dots, q-1$ qiymatlarni, faqat bir marta qabul qiladi. Bunda $q > 4\Lambda + 1$;

$$\text{Va } v(s) - \Lambda \leq v(s) + \theta_s \Lambda \leq v(s) + \Lambda,$$

$$2\Lambda \leq v(s) \leq q - 2\Lambda \quad (2.1.9)$$

Shartni qanoatlantiruvchi $v(s)$ lar uchun quyidagi ifoda o‘rinli bo‘ladi

$$(R\alpha + as) \geq \min \left(\frac{v(s) - \Lambda}{q}, \frac{q - v(s) - \Lambda}{q} \right) \geq \min \left(\frac{v(s)}{2q}, \frac{q - v(s)}{2q} \right).$$

Shuning uchun L yig‘indining har bir ifodasini, (2.1.9) ifodadagi shartni qanoatlantiruvchi $v(s)$ qiymatga almashtirishimiz mumkin

$$\max \left(\frac{2q}{v(s)}, \frac{2q}{q - v(s)} \right);$$

Endi L yig‘indining har bir ifodasini, shartni qanoatlantirmaydigan qiymatga almashtiramiz. Chunki ko‘rib chiqilayotgan yig‘indining shartni qanoatlantirmaydigan $v(s)$ shartlari ham mavjud bo‘ladi. Bu shartlar $4\Lambda + 1$ dan ortiq emas va almashtirishdan so‘ng quyidagi ifodaga ega bo‘lamiz,

$$L \leq (4\Lambda + 1)P + 2q \sum \max\left(\frac{2q}{v(s)}, \frac{2q}{q - v(s)}\right);$$

bu yerda $v(s)$ (2.1.9) ifodani qanoatlantiruvchi barcha qiymatlarga tegishlidir.

Bu yerdan esa biz

$$L \leq (4\Lambda + 1)P + 2q \cdot 2 \sum_{v=2}^{\lfloor \frac{q}{2} \rfloor} \frac{1}{v} = O(\Lambda P + q \log q)$$

ifodaga ega bolamiz.

Oxirgi munosabat shubhasiz o‘rinlidir. Bunda $q \leq 4\Lambda + 1$ bo‘lganda har bir L yig‘indi uchun qiymat R ya’ni haqiqiy qiymatga teng bo‘lmagan holatda va

$$(2.1.5) \text{ ifodaning o‘ng tomonini } \theta = \left(\frac{P^{n-1}}{q} + 1\right) \text{ ga teng qiymatidan}$$

quyidagini topamiz,

$$|S|^{2^{n-1}} \leq P^{2^{n-1}-1} + P^{2^{n-1}-n-\varepsilon_1} (\Lambda P + q \log q) \left(\frac{P^{n-1}}{q} + 1\right)$$

yoki

$$|S|^{2^{n-1}} \leq P^{2^{n-1}-1} + P^{2^{n-1}-n+\varepsilon_1} \left(\Lambda + \frac{q \log q}{P}\right) \left(\frac{1}{q} + \frac{1}{P^{n-1}}\right) \quad (44)$$

Yana shuni ta’kidlash lozimki, biz $q < P^n$ shart uchun ko‘rib chiqishimiz lozim, chunki ifodaning o‘ng qismida $P^n P^{2^{n-1}} \dots$; shuning uchun biz $\log q = O(P)$ ga egamiz, bu yerda ε_2 ixtiyoriy noldan katta musbat son. Ushbu $\varepsilon_1 + \varepsilon_2 = 2n - 1$ tenglikdan quyidagiga ega bo‘lamiz,

$$|S|^{2^{n-1}} \leq P^{2^{n-1}(1+\varepsilon)} \left(\Lambda + \frac{q}{P}\right) \left(\frac{1}{q} + \frac{1}{P^{n-1}}\right)$$

yoki yakuniy

$$|S| \leq P^{(1+\varepsilon)} \left(\left(\Lambda + \frac{q}{P}\right) \left(\frac{1}{q} + \frac{1}{P^{n-1}}\right)\right)^{2^{-n+1}} \quad (2.1.10)$$

ga ega bo‘lamiz.

Veyl yig'indisi uchun topilgan taxmin faqat λ tartibi q tartibidan past bo'lganda foydali bo'ladi. Shu bilan birga bu ifoda ko'pincha $\lambda = 1$ qiymat uchun ishlatiladi. Bu holda q ning qiymati ahamiyatsiz emas. q ning tartibi yuqori bo'lsa P^ε dan past bo'lsa, 1935 yilda I.M. Vinogradov Veyl yig'indisi uchun yangi usulni topdi, u $n > 13$ bo'lganda maksimal chegarani beradi.

(2.1.10) ifodaga asosan maxrajdagi q qiymatni quyidagicha ifodalashimiz mumkin,

$$P^{1-\eta} < q \leq cP^{\eta-1}$$

Qachonki $0 < \eta < 1$ va qiymat > 1 doimiy son bo'ladi. $L = 1$ bo'lgan holatida ifoda quyidagi ko'rinishga keladi,

$$S \ll P^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{P} + \frac{q}{P^n} \right)^{2^{-n+1}}$$

Agar,

$$P^{1-\eta} < q \leq P,$$

bo'lsa,

$$S \ll P^{1+\varepsilon} \cdot P^{-(1-\eta)2^{-n+1}},$$

bo'ladi yoki,

$$S \ll P^{1-2^{-n+1}+\eta 2^{-n+1}+\varepsilon}.$$

Agar,

$$P < q \leq P^{\eta-1},$$

bo'lsa,

$$S \ll P^{1+\varepsilon} \cdot P^{-2^{-n+1}} \ll P^{1-2^{-n+1}+\varepsilon},$$

bo'ladi,

$$P^{\eta-1} < q \leq cP^{\eta-1},$$

bo'lsa,

$$S \ll P^{1+\varepsilon} \left(\frac{P^{\eta-1}}{P^n} \right)^{2^{-n+1}} \ll P^{1-2^{-n+1}+\varepsilon},$$

bo'ladi.

Shunday qilib bizning ifodamizda har doim

$$S \ll P^{1-2^{-n+1}+\eta^{2^{-n+1}+\varepsilon}}$$

o‘rinli bo‘ladi deb taxmin qilamiz.

Veyl usulidan foydalanib Lo-Ken Xua (1938) lemmasini isbotlaymiz, bu bizga muhim va bir-biridan qiziqarli masalalarni yechishda yordam beradi. Avval Xua Lo-Ken lemmasi I.M. Vinogradov tomonidan isbotlangan. Shu bilan birga I.M. Vinogradov lemmasi $n > 13$ holat uchun Xua Lo-Ken lemmasiga qaraganda aniqroq, $n \leq 13$ holat uchun aniqligi nisbatan pastroq hisoblanadi.

Xua Lo-Ken lemmasi. Noaniq tenglamaning K yechimlari soni

$$x_1^n + \dots + x_s^n = x_{s+1}^n + \dots + x_{2s}^n$$

bunda x_1, \dots, x_{2s} , miqdorlar

$$1 \leq x_1 \leq P, \dots, 1 \leq x_{2s} \leq P,$$

tengsizlikni qanoatlantiradi va $s = 2^{k-1}$ ($k = 1, 2, \dots, n$) holatida

$$K = O(P^{2^k - k + \varepsilon}),$$

ifoda o‘rinli bo‘ladi.

$k = 1$ holat uchun lemma natijasi ma‘lum. Buni to‘liq induksiya usulida isbotlaymiz. Buning uchun unga analitik ifodani berish foydalidir.

Ma‘lumki, agar z faqat butun qiymatlar qabul qilsa, u holda

$$\int_0^1 e^{2\pi iz\alpha} d\alpha = z \begin{cases} 1, & \text{agar } z = 0 \\ 0, & \text{agar } z \neq 0 \end{cases} \quad \text{o‘rinli bo‘ladi.}$$

Shuning uchun buni tekshirish oson, chunki

$$K = \sum_{x_1=1}^P \dots \sum_{x_{2s}=1}^P e^{2\pi i(x_1^n + \dots + x_s^n - x_{s+1}^n - \dots - x_{2s}^n)\alpha} d\alpha$$

yoki,

$$K = \int_0^1 |S(\alpha)|^{2s} d\alpha,$$

bunda

$$S(\alpha) = \sum_{x=1}^P e^{(2\pi i\alpha x^n)}$$

ifodaga teng bo'ladi.

Bundan kelib chiqib ko'rib chiqilayotgan lemma quyidagi munosabatga ekvivalent degan hulosaga kelamiz

$$\int_0^1 |S(\alpha)|^{2k} d\alpha = O(P^{2k-k+\varepsilon})$$

bunda $k = 1, 2, \dots, n$ bo'lishi lozim.

Vinogradov lemmasi. Agar $P \geq 1$, $n > 1$, $\nu = \frac{1}{n}$, z – haqiqiy son bo'lsa,

u holda integral

$$J = \int_0^1 e^{2\pi izy^n} dy,$$

ko'rinishga ega bo'ladi va bunda,

$$|J| \leq Z,$$

qachonki,

$$Z = \begin{cases} P, & \text{agar } |z| \leq P^{-n} \\ \sqrt{2}|z|^{-\nu}, & \text{agar } |z| > P^{-n} \end{cases},$$

ifoda bajarilganda o'rinli bo'ladi. Buning uchun $|z| > P^{-n}$ ko'rib chiqish kifoya, Z bunda istalgan son bo'lishi mumkin. shu bilan birga $2zy^n = u$ deb belgilash kiritib olsak, u holda,

$$J = U + iV; \quad U = \int_0^T \varphi(u) \cos \pi u du, \quad V = \int_0^T \varphi(u) \sin \pi u du,$$

$$T = 2zP^n, \quad \varphi(u) = \frac{\nu u^{\nu-1}}{(2z)^\nu}; \quad T > 2,$$

keling U ni integral yig'indisiga quyidagicha ajratamiz,

$$U = \int_0^{\frac{1}{2}} + S, \quad S = \int_{\frac{1}{2}}^{\frac{3}{2}} + \int_{\frac{3}{2}}^{\frac{5}{2}} + \dots + \int_{\left[\frac{r+1}{2}\right] - \frac{1}{2}}^{\frac{r}{2}}$$

yig'indi S ning k -hadi ko'rinishiga ega

$$\int_{k-\frac{1}{2}}^{k-\frac{1}{2}+t} \varphi(u) \cos \pi u du.$$

Bundan tashqari yig'indi S ning oxirgi t hadi uchun $0 \leq t < 1$, qolgan barcha hadlari uchun esa $t=1$ tenglik o'rinli bo'ladi.

Keling almashtirish kiritamiz, bunda

$$u = k - \frac{1}{2} + v,$$

bo'lib, yig'indi S ning k -hadi ko'rinishi quyidagicha bo'ladi,

$$\int_0^t \varphi\left(k - \frac{1}{2} + v\right) \cos \pi \left(k - \frac{1}{2} + v\right) dv = (-1)^k \int_0^t \varphi\left(k - \frac{1}{2} + v\right) \sin \pi v dv.$$

$\varphi(u)$ funksiya kamayuvchi bo'lganligi sababli, oxirgi tenglikdan xulosa qilamizki, S yig'indining hadlari ularning mutlaq qiymatidagi son ortishi bilan kamayadi. S yig'indisi shartlarining belgilari almashishini ham ko'rib, shunday hulosaga kelamiz,

$$|S| < \left| \int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(u) \cos \pi u du \right|,$$

shuning uchun,

$$U \leq \int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(u) \cos \pi u du + |S| < \int_0^{\frac{1}{2}} \varphi(u) \cos \pi u du + \left| \int_{\frac{1}{2}}^{\frac{3}{2}} \varphi(u) \cos \pi u du \right| \ll \ll \int_0^{\frac{1}{2}} \varphi(u) du = \left[\left(\frac{u}{2z} \right)^v \right]_0^{\frac{1}{2}} = z^{-v}.$$

V ni integrallar yig'indisi shakliga keltirilsa,

$$V = \int_0^1 + \int_1^2 + \dots + \int_{|r|}^r ,$$

biz ham huddi shunday qabul qilamiz

$$|V| < z^{-\nu}$$

va

$$|J| = \sqrt{U^2 + V^2} < \sqrt{z^{-2\nu} + z^{-2\nu}} = \sqrt{2}z^{-\nu},$$

yakuniy ko‘rinishga keladi.

2.2-§. Veyl teoremasi tipidagi teoremlarning Varing masalasiga tatbiqi.

$$f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_1 x + \alpha_0 - k, \quad (k \geq 2) \text{ darajali haqiqiy}$$

$\alpha_k = \alpha, \alpha_{k-1}, \dots, \alpha_1, \alpha_0$ ko‘yfisentli ko‘phad va

$$S(f) = \sum_{n \leq P} e(f(n)) \quad (2.2.1)$$

bo‘lsin. U holda Veyl tengsizligiga ko‘ra $(a, q) = 1, |\alpha - aq^{-1}| \leq q^{-2}$

bo‘lganda

$$S(f) \ll P^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}} \quad (2.2.2)$$

bajariladi (R. Von [13] dagi 2.4-lemma). Bunda ε ixtiyoriy musbat son, Vinogradov simvoli \ll dagi doimiy k va $\varepsilon > 0$ ga bog‘liq.

2.2.1-teorema. Agar $k > 6, \alpha = aq^{-1} + z, (a, q) = 1, |z| < q^{-2}$ va $\alpha_{k-1} = 0$

bo‘lsa, u holda

$$S(f) \ll P^{1+\varepsilon} \left(Pz_0^{-1} q^{-1} + P^{-2} + qz_0 P^{-k} \right)^{\frac{4}{3} 2^{-k}}, \quad (2.2.3)$$

bajariladi. Bunda $z_0 = \max(1; P^k |z|)$.

Bu yerdan, xususiyl holda

$$S(f) \ll P^{1+\varepsilon} \left(q^{-1} + P^{-1} + qP^{-k} \right)^{\frac{4}{3} 2^{-k}}, \quad (2.2.4)$$

ekanligi kelib chiqadi

(2.2.4) baho $P^3 \leq q \leq P^{k-3}$ bo‘lganda (2.2.2) dan yaxshi. Shuning uchun ham (2.2.3) baho Veyl bahosi (2.2.2) dan kuchliroq. Shuningdek, 2.1.1-teorema ilgari

I. M. Vinogradov tomonidan $S(f)$ yig'indi uchun olingan bahodan $6 \leq k \leq 11$ bo'lganda kuchlidir.

Bu teoremani $\{f(n)\}$ ketma-ketlikning kasr qismlarining taqsimlanishini baholashga va sonlar nazariyasining ba'zi bir trigonometrik masalalarini tekshirishga tatbiq etish mumkin.

Xususan (2.2.3) bahodan $k \geq 6$ bo'lganda

$$\|f(n)\| \leq n^{\varepsilon - \frac{8}{3}2^{-k}}.$$

Tengsizlikning cheksiz ko'p natural yechimlarga ega ekanligi kelib chiqadi (I.Allakov[14,15]).

Bu natijalar B. R. Heath-Brown [16] natijalarining ko'phadlar uchun umumlashtirilganidir.

2.2.1-teoremaning isboti. $f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_1 x + \alpha_0$ ixtiyoriy $k \geq 6$ darajali haqiqiy koeffitsiyentli ko'phad bo'lsin. (2.2.3) bahoni isbotlash uchun avvalo

$$\Delta_h(f(x)) = f(x+h) - f(x-h),$$

dan foydalanib matematik induksiya metodi bilan ixtiyoriy $j, 0 \leq j \leq k$ natural soni uchun

$$|S(f)|^{2^j} \leq 2^{2^j-1} P^{2^j-j-1} \sum_{|h_1| < P/2} \dots \sum_{|h_j| < P/2} \left| \sum_{n \in I_j} e(\Delta_j(n)) \right|. \quad (2.2.5)$$

Tengsizlikning o'rinli ekanligini ko'rsatamiz. Bu yerda

$$I_1 = I(h_1) \subset [1; P], I_j = I(h_1, \dots, h_j) \subset I_{j-1} = I(h_1, \dots, h_{j-1}) \text{ va}$$

$$\Delta_j(n) = \Delta_{h_1} \dots \Delta_{h_j}(f(n)).$$

$j=0$ bo'lganda (2.2.5) tengsizlik o'z-o'zidan tushunarli. Endi faraz etaylik, (2.2.5) tengsizlik biror j uchun o'rinli bo'lsin, u holda uning $j+1$ uchun ham o'rinli ekanligini ko'rsatamiz.

$$\left| \sum_{n \in I_j} e(\Delta_j(n)) \right| \leq \sum_{i=0,1} \left| \sum_{\substack{n \in I_j \\ n \equiv i \pmod{2}}} e(\Delta_j(n)) \right|.$$

Bo'lganligi sababli Koshi tengsizligidan

$$\left| \sum_{i=1}^N X_i \right|^m \leq N^{m-1} \sum_{i=1}^N |X_i|^m.$$

Ko'rinishida foydalanib quyidagi tengsizlikka ega bo'lamiz:

$$\begin{aligned} |S(f)|^{2^{j+1}} &\leq 2^{2^{j+1}-2} P^{2^{j+1}-2j-2} \left(\sum_{|h_1| < P/2} \dots \sum_{|h_j| < P/2} \sum_{i=0,1} \left| \sum_{\substack{n \in I_j \\ n \equiv i \pmod{2}}} e(\Delta_j(n)) \right| \right)^2 \leq \\ &\leq 2^{2^{j+1}-2} P^{2^{j+1}-2j-2} \times P^j \times 2 \sum_{|h_1| < P/2} \dots \sum_{|h_j| < P/2} \sum_{i=0,1} \left| \sum_{\substack{n \in I_j \\ n \equiv i \pmod{2}}} e(\Delta_j(n)) \right|^2. \end{aligned} \quad (2.2.6)$$

Bu yerda

$$\begin{aligned} \sum_{i=0,1} \left\| \sum_{\substack{n \in I_j \\ n \equiv i \pmod{2}}} e(\Delta_j(n)) \right\|^2 &= \sum_{i=0,1} \sum_{\substack{m, n \in I_j \\ m \equiv i \pmod{2}, n \equiv i \pmod{2}}} e(\Delta_j(m) - \Delta_j(n)) = \\ &= \sum_{\substack{m, n \in I_j \\ m \equiv n \pmod{2}}} e(\Delta_j(m) - \Delta_j(n)) = \sum_{|h_j| < P/2} \sum_{r \in I_{j+1}} e(\Delta_j(r+h) - \Delta_j(r-h)) = \\ &= \sum_{|h_j| < P/2} \sum_{r \in I_{j+1}} e(\Delta_{j+1}(r)), \quad I_{j+1} = \{r; r \pm h \in I_j\}. \end{aligned} \quad (2.2.7)$$

(2.2.6) va (2.2.7) lardan

$$|S(f)|^{2^{j+1}} \leq 2^{2^{j+1}-1} P^{2^{j+1}-j-2} \left(\sum_{|h_j| < P/2} \dots \sum_{|h_j| < P/2} \sum_{|h_{j+1}| < P/2} \left| \sum_{r \in I_{j+1}} e(\Delta_{j+1}(r)) \right| \right).$$

Ekanligi ya'ni (2.2.5) tengsizlikning $j+1$ uchun o'rinli ekanligi kelib chiqadi.

Shunday qilib, (2.2.5) tengsizlik ixtiyoriy j natural soni uchun o'rinli ekan.

Endi, (2.2.5) tengsizlikda $j = k - 3$ deb olib,

$$\Delta_{k-3}(n) = \Delta_{h_1} \dots \Delta_{h_{k-3}}(f(n)),$$

ni qaraymiz. Bu yerda

$$\begin{aligned} (x+h)^k - (x-h)^k &= 2h(C_k^1 x^{k-1} + C_k^3 x^{k-3} h^2 + C_k^5 x^{k-5} h^4 + \dots) = \\ &= 2h \left(kx^{k-1} + \sum_{1 \leq l \leq \frac{1}{2}(k-1)} C_k^{2l+1} x^{k-2l-1} h^{2l} \right), \end{aligned}$$

va

$$C_m^i = \frac{m!}{i!(m-i)!},$$

ekanligini e'tiborga olsak,

$$\begin{aligned} \Delta_h(f(x)) &= \alpha \cdot 2h \left(kx^{k-1} + \sum_{1 \leq l \leq \frac{1}{2}(k-1)} C_k^{2l+1} x^{k-2l-1} h^{2l} \right) + \\ &+ \alpha_{k-1} \cdot 2h \left((k-1)x^{k-2} + \sum_{1 \leq l \leq \frac{1}{2}(k-2)} C_k^{2l+1} x^{k-2l-2} h^{2l} \right) + \\ &+ \alpha_{k-2} \cdot 2h \left((k-2)x^{k-3} + \sum_{1 \leq l \leq \frac{1}{2}(k-3)} C_k^{2l+1} x^{k-2l-3} h^{2l} \right) + \\ &+ \dots + \alpha_2 \cdot 4hx + \alpha_1 \cdot 2h, \end{aligned}$$

bo'ladi. Shuning uchun ham

$$\begin{aligned} \Delta_{h_1} \dots \Delta_{h_{k-3}}(f(x)) &= 2^{k-3} h_1 \dots h_{k-3} \left\{ \alpha \left(\frac{1}{6} k! x^3 + \alpha x \right) + \right. \\ &+ \alpha_{k-1} \left(\frac{1}{2} (k-1)! x^2 + b \right) + \alpha_{k-2} (k-2)! x + \alpha_{k-3} c \} = \\ &= 2^{k-3} h_1 \dots h_{k-3} \left\{ \frac{1}{6} k! \alpha_k x^3 + \frac{1}{2} (k-1)! \alpha_{k-1} x^2 + \right. \\ &+ (\alpha_k \alpha + \alpha_{k-2} (k-2)!) x + (\alpha_{k-1} b + \alpha_{k-3} c) \}, \end{aligned} \tag{2.2.8}$$

deb yoza olamiz. Bunda a, b, c -butun sonlar

(2.2.5) va (2.2.8) lardan

$$|S(f)|^{2^{j+1}} \ll P^{2^{k-3}-k+2} \sum_{|h_1| < P/2} \dots \sum_{|h_{k-3}| < P/2} \left| \sum_{n \in i_{k-3}} e(2^{k-3} \times \right.$$

$$\times h_1 \dots h_{k-3} \left(\frac{1}{6} k! \alpha n^3 + \frac{1}{2} (k-1)! \alpha_{k-1} n^2 + (\alpha_k \alpha + \alpha_{k-2} (k-2)! n) \right), \quad (2.2.9)$$

kelib chiqadi. Bundan keyin (2.2.9) bahoda $\alpha_{k-1} = 0$ deb olamiz.

U holda

$$\begin{aligned} |S(f)|^{2^{k-3}} &<< P^{2^{k-3}-k+2} \sum_{|h_1| < P/2} \dots \sum_{|h_{k-3}| < P/2} \left| \sum_{n \in I_{k-3}} e(2^{k-3} n) \right| \times \\ &\times h_1 \dots h_{k-3} \left(\frac{1}{6} k! \alpha n^3 + (\alpha_k \alpha + \alpha_{k-2} (k-2)! n) \right), \end{aligned} \quad (2.2.10)$$

hosil bo'ladim m -butun soni uchun $L(m) = [mP^{-3}; (m+1)P^{-3}]$ va x haqiqiy soni uchun $x \in L(m)$ bajariladigan butun sonni $m = m(x)$ deb belgilaymiz. Shuningdek, $L(x) = L(m(x))$ va

$$T(x) = \max_{y, z} \left| \sum_{n \in I} e(yn^3 + zn) \right|.$$

Belgilashlarni kiritamiz. Bunda I bilan $[1; P]$ intervalning biror qism intervali belgilangan, y o'zgaruvchi $L(m)$ da, z esa $[0; 1]$ intervalda o'zgaradi.

Bu yangi belgilashlardan foydalanib (2.2.10) dan

$$|S(f)|^{2^{j+1}} << P^{2^{k-3}-k+2} \sum_{|h_1| < P/2} \dots \sum_{|h_{k-3}| < P/2} T(\alpha h_1 \dots h_{k-3}), \quad K = \frac{1}{6} 2^{k-3} k!, \quad (2.2.11)$$

ni hosil qilamiz.

Endi $h = Kh_1 h_2 \dots h_{k-3}$ deb olib va (2.1.11) ning o'ng tomonida $h = 0$ hadlarning hisyasi $<< P^{k-4}$ bo'lganligi hamda h ni yuqoridagi ko'paytma ko'rinishida ifodalashlar soni $<< \tau_{k-2}(h) << h^\varepsilon << P^\varepsilon$ ekanligini e'tiborga olsak, (2.2.11) dan

$$|S(f)|^{2^{k-3}} << P^{2^{k-3}-1} + P^{2^{k-3}-k+2+\varepsilon} \sum_{h=1}^R T(\alpha h), \quad R = KP^{k-3},$$

ga ega bo'lamiz. Bu bahoning ikkala tomonini 6-darajaga ko'tarib Gyolder tengsizligini qo'llasak, quyidagi bahoga ega bo'lamiz.

$$\begin{aligned}
|S(f)|^{3-2^{k-2}} &<< P^{3 \cdot 2^{k-2}-6} + P^{3 \cdot 2^{k-2}-6k+12+6\varepsilon} \left(\sum_{h=1}^R T(\alpha h) \right)^6 << \\
&<< P^{3 \cdot 2^{k-2}-6} + P^{3 \cdot 2^{k-2}-6k+12+6\varepsilon} R^{6-1} \sum_{h=1}^R T^6(\alpha h) << P^{3 \cdot 2^{k-2}-6} + P^{3 \cdot 2^{k-2}-k-3+6\varepsilon} \sum_{h=1}^R T^6(\alpha h).
\end{aligned}$$

(2.2.12)

(2.2.12) ning o‘ng tomonidagi yig‘indini baholash uchun D.R.Heath-Brown[16] dagi quyidagi lemmalardan foydalanamiz :

2.2.1-lemma. Ixtiyoriy $\varepsilon > 0$ soni uchun

$$\sum_{*1}^* T(mP^{-3})^6 << P^{7+\varepsilon}$$

baho o‘rinli.

2.2.2-lemma. Agar $\alpha = aq^{-1} + z$, $(a, q) = 1$ va $|z| \leq q^{-2}$ bo‘lsa, u holda

ixtiyoriy haqiqiy μ va $0 < \Delta < \frac{1}{2}$ sonlari uchun

$$\{h \leq H : \|\alpha h - \mu\| \leq \Delta\} \leq 4(1 + q\Delta)(1 + q^{-1}H) \quad (2.2.13)$$

va

$$\{h \leq H : \|\alpha h - \mu\| \leq \Delta\} \leq 8(1 + q|z|^{-1}\Delta)(1 + q|z|H) \quad (2.2.14)$$

tengsizliklar o‘rinli.

Bular mos ravishda D.R. Heath-Brown [16] dagi 5 va 6- lemmalardir.

Endi, 1-lemmani qo‘llab (2.2.12) dan

$$|S(f)|^{3-2^{k-2}} << P^{3 \cdot 2^{k-3}-6} + P^{3 \cdot 2^{k-3}-k+4+7\varepsilon} \max_m F(m), \quad (2.2.15)$$

ni hosil qilamiz. Bunda

$$F(m) = \{h \leq P : \alpha h \in \bigcup_{i=-\infty}^{\infty} L(m + P^3 l)\} \leq \{h \leq R : \|\alpha h - mP^{-3}\| \leq P^{-3}\},$$

$H = R = KP^{k-3}$, $\Delta = P^{-3}$ bo‘lganda (2.1.13) va (2.2.14) lardan mos ravishda

$$F(m) << qP^{-3} + P^{k-4} + q^{-1}P^{k-3}$$

va

$$F(m) \ll q|z|P^{k-3} + P^{k-6} + q^{-1}|z|^{-1}P^{-3}.$$

Kelib chiqadi. $|z_0| = \max\{1, |z|P^k\}$ deb olib bu ikkala bahoni birlashtirsak,

$$F(m) \ll qz_0P^{-3} + P^{k-6} + q^{-1}z_0P^{k-3}, \quad (2.2.16)$$

hosil bo'ladi. Nihoyat, (2.2.15) va (2.2.16) lardan

$$\begin{aligned} |S(f)|^{3 \cdot 2^{k-2}} &\ll P^{3 \cdot 2^{k-2} - 6} + qzP^{3 \cdot 2^{k-2} - k + 1 + 7\varepsilon} + P^{3 \cdot 2^{k-2} - 2 + 7\varepsilon} + q^{-1}z_0^{-1}P^{3 \cdot 2^{k-2} + 1 + 7\varepsilon} \ll \\ &\ll P^{3 \cdot 2^{k-2} + \varepsilon} \left(qz_0P^{-k+1} + P^{-2} + q^{-1}z_0^{-1}P \right)^{\frac{1}{3 \cdot 2^{k-2}}}. \end{aligned}$$

Ya'ni

$$|S(f)| \ll P^{2^{1+\varepsilon}} \left(qz_0P^{-k+1} + P^{-2} + q^{-1}z_0^{-1}P \right)^{\frac{1}{3 \cdot 2^{k-2}}},$$

bahoga ega bo'lamiz.

Shuni ham ta'kidlash kerakki, k – darajali, haqiqiy $\alpha_k = \alpha, \alpha_{k-1}, \dots, \alpha_1, \alpha_0$

ko'effitsiyentli ixtiyoriy

$$f(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_1 x + \alpha_0,$$

ko'phadni

$$x = y - \alpha_{k-1} (k\alpha_k)^{-1},$$

almashtirish yordamida

$$f(y) = \beta_k y^k + \beta_{k-2} y^{k-2} + \beta_{k-3} y^{k-3} + \dots + \beta_1 y + \beta_0.$$

Ko'rinishiga keltirish mumkin. Lekin bu almashtirish hamma vaqt ham butun sonlar to'plamidagi almashtirish bo'lavermaydi.

2.3-§. Ketma ketliklar kasr qismlarining taqsimlanishini baholashda tatbiqi.

$\alpha, \beta, \gamma, \delta$ -haqiqiy sonlar, $\{\theta\}$ esa θ ning kasr qismi, $\|x\|$ bilan x dan unga eng yaqin turgan butun songacha bo'lgan masofani, $\pi_D(N)$ bilan esa $p \leq N$ va $p \equiv f \pmod{D}$ shartlarni qanoatlantiruvchi tub sonlarning sonini belgilaymiz.

2.3.1-teorema. Agar $0 \leq \gamma < \gamma + \delta \leq 1$ va $\pi_D^*(\gamma, \delta, N)$ esa $p \leq N$ va $p \equiv f \pmod{D}$, $\gamma \leq \{\alpha p^k\} < \gamma + \delta$ shartlarni qanoatlantiruvchi tub sonlarning soni bo'lsa, u holda

$$\pi_D^*(\gamma, \delta, N) - \delta \pi_D(N) \ll \frac{N}{D} \left(\Delta(\delta^{-1}) \ln q + \left(\frac{qN}{\delta} \right)^\varepsilon \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}} \right),$$

bo'ladi.

2.3.2-teorema. Agar $k \geq 2$ natural son α -irratsional son va β esa ixtiyoriy haqiqiy son bo'lsa, u holda ixtiyoriy $\varepsilon > 0$ soni uchun faqat k va ε larga bog'liq bo'lgan musbat $C(\varepsilon, k)$ mavjudki, $\nu = 4^{1-k}$, $D \ll N^\varepsilon$ bo'lganda

$$\|\alpha p^k - \beta\| \leq C(\varepsilon, k) p^{-\frac{\nu}{2} + \varepsilon},$$

tengsizlik $p \equiv f \pmod{D}$ tub sonlarda cheksiz ko'p yechimga ega bo'ladi.

Bu teoremlar A. Ghosh [17] natijalarining k va D bo'yicha umumlashtirilgani, I.M. Vinogradov natijalarining $2 \leq k \leq 11$ bo'lgandagi kuchaytirilganidir. Xususiyl holda $k = 2$ va $D = 1$ bo'lganda 2.3.1 va 2.3.2-teoremlardan mos ravishda A. Ghoshdagi 3 va 4 -teoremlar kelib chiqadi.

Shuni ham ta'kidlash kerakki, ushbu paragrafda foydalanilgan metod, nafaqat αp^k ga nisbatan 2.3.1 va 2.3.2-teoremlarni isbotlash imkonini beribgina qolmay, balki $f(x) = \alpha p^k + \alpha_1 p^{k-1} + \dots + \alpha_{k-1} p + \alpha_k$ ko'phad uchun ham shunday teoremlarni isbotlash imkonini beradi.

η , $0 \leq \eta \leq \frac{1}{2}$ shartni qanoatlantiruvchi haqiqiy son va

$$g_\eta(x) = \begin{cases} 1, & \text{agar } -\eta \leq x < \eta \text{ bo'lsa} \\ 0, & \text{agar } -\frac{1}{2} \leq x < -\eta \text{ yoki } \eta \leq x < \frac{1}{2} \text{ bo'lsa} \end{cases}, \quad (2.3.1)$$

bo'lsin. 2.3.1-teoremani isbotlashda quyidagi lemmadan foydalanamiz.

2.3.1-lemma. Agar α soni $|\alpha q - a| \leq q^{-1}$, $(a, q) = 1$ shartni qanoatlantiruvchi irratsional son, β esa ixtiyoriy haqqiy son bo'lsa, u holda ixtiyoriy $\varepsilon_0 > 0$ soni uchun

$$\sum_{n \leq N, n \neq f} \Lambda(n) \{g(\alpha n^k - \beta) - 2\eta\} \ll \frac{N}{D} \left[\Delta(\eta^{-1}) \ln(2Nq) + (Nq\eta^{-1})^{\varepsilon_0} \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}} \right].$$

Baho o'rinli. Bu yerda $\Delta(\eta^{-1})$ (2.3.1) tenglik yordamida aniqlanadi va $R = 2^{k-1}$

Isboti. $g_\eta(x)$ funksiyani x bo'yicha davriy (davri 1 ga teng) davom ettirib aniqlanish sohasini kengaytiramiz. $g_\eta(x)$ ni Fure qatoriga yoyib

$$g_\eta(x) - 2\eta = \sum_{\substack{m \neq 0 \\ -\infty}}^{\infty} \frac{\sin 2\pi m \eta}{\pi m} e(mx). \quad (2.3.2)$$

ni hosil qilamiz. Bu yerda \sum' simvol m bo'yicha yig'indida $\pm m$ ga mos hadlarning birlashtirilishini bildiradi.

Endi α -irratsional son, β esa ixtiyoriy haqqiy son bo'lsin. $x = \alpha n^k - \beta$ deb olib, (2.3.1) dan

$$g_\eta(\alpha n^k - \beta) - 2\eta = \sum_{\substack{m \neq 0 \\ -\infty}}^{\infty} \frac{\sin 2\pi m \eta}{\pi m} e(m(\alpha n^k - \beta)), \quad (2.3.3)$$

ni hosil qilamiz.

Agar $\frac{a}{q}$ kasr $|\alpha q - a| \leq q^{-1}$, $(a, q) = 1$ shartlarni qanoatlantiruvchi α sonining uzluksiz kasrga yoyilmasi va $H = [Nq\eta^{-1}] + 1$ bo'lsin, u holda (2.3.3) dan

$$\sum_{n \leq N, n \neq f} \Lambda(n) \{g(\alpha n^k - \beta) - 2\eta\} = A_1 + A_2 \quad (2.3.4)$$

kelib chiqadi. Bu yerda

$$A_1 = \sum_{|m| \leq H} \frac{\sin 2\pi m \eta}{\pi m} e(-\beta m) \sum_{\substack{n \leq N \\ n \neq f}} \Lambda(n) e(mn^k \alpha),$$

$$A_2 = \sum_{\substack{n \leq N \\ n \neq f}} \Lambda(n) \sum_{|m| > H} \frac{\sin 2\pi m \eta}{\pi m} e(m(\alpha n^k - \beta)).$$

Avvalo A_2 yig'indini qaraymiz. Uni

$$A_2 = \frac{1}{2i} \sum_{\substack{n \leq N \\ n \neq f}} \Lambda(n) \sum \frac{1}{m\pi} \left(e\left(m(\alpha n^k - \beta + \eta)\right) - e\left(m(\alpha n^k - \beta - \eta)\right) \right). \quad (2.3.5)$$

Ko'rinishida yozib olishimiz mumkin. Qulaylik uchun $\theta_n^\pm = \alpha n^k - \beta \pm \eta$

va

$$T_n^\pm(u) = \sum_{m \leq u} e(m\theta_n^\pm),$$

deb belgilab olamiz.

Faraz qilaylik, θ_n^\pm butun son bo'lmasin, (aks holda uning (2.3.5) dagi hissasi nolga teng bo'ladi). $D = 1$ da

$$T_n^\pm(u) \leq \frac{1}{2} \|\theta_n^\pm\|^{-1}$$

hosil bo'ladi. Shuning uchun ham Abelning yig'ish formulasidan foydalansak, bu yerdan

$$\left| \sum_{|m| > H} \frac{1}{m\pi} e(\theta_n^\pm) \right| \ll H^{-1} \|\theta_n^\pm\|^{-1},$$

ga ega bo'lamiz.

Ikkinchi tomondan esa bu yig'indi uchun

$$\left| \sum_{|m| > H} \frac{1}{m\pi} e(\theta_n^\pm) \right| \ll 1,$$

baho o'rinli. Shuning uchun ham (2.3.4) dan

$$A_2 \ll (\ln N) \sum_{n \leq N, n \neq f} \min(1; H^{-1} \|\alpha n^k - \beta \pm \eta\|), \quad (2.3.6)$$

kelib chiqadi. (2.3.6) bahoning o'ng tomonidagi yig'indini baholash uchun

$\Phi(H, \omega) = \min(1; H^{-1} \|\omega\|^{-1})$ funksiyani qaraymiz. Tushunarliki, $\Phi(H, \omega)$

funksiya ω bo'yicha davriy funksiya bo'lib davri 1 ga teng. Bu funksiyani Fure qatoriga yoysak,

$$\Phi(H, \omega) = \sum_{-\infty}^{\infty} a_h e(h\omega), \quad (2.3.7)$$

hosil bo'ladi. Bu yerda $a_h \ll \min((1 + \ln H)H^{-1}; |h|^{-1}; Hh^{-2})$.

(2.3.7) tenglikni

$$\Phi(H, \omega) = \sum_{|h| \leq H^2} a_h e(h\omega) + \sum_{|h| > H^2} a_h e(h\omega) + O\left(\frac{\ln H}{H}\right),$$

ko'rinishda yozish mumkin. (2.3.6) ga asosan bu yerdan

$$A_2 = B_1 + B_2 + O(N(DH)^{-1} \ln^2 HN) \quad (2.3.8)$$

ni hosil qilamiz. Bunda

$$B_1 \ll (\ln NH)^2 \sum_{h \leq H^2} \min(H^{-1}; |h|^{-1}) \left| \sum_{n \leq N, n \neq f} e(hn^k \alpha) \right|,$$

$$B_2 \ll (\ln N) \sum_{n \leq N, n \neq f} \left| \sum_{|h| > H^2} a_h e(h(n^k \alpha - \beta \pm \eta)) \right|,$$

B_2 dagi yig'indilarni baholashda davom etib,

$$B_2 \ll (\ln N) \cdot H \sum_{n \leq N, n \neq f} \sum_{|h| > H^2} h^{-2} \ll \frac{HN}{D} (\ln N) \int_{H^2}^{\infty} \xi^{-2} d\xi \ll \frac{N}{DH} \ln N,$$

ga ega bo'lamiz.

Endi B_1 ni qaraymiz. B_1 uchun yuqorida olingan bahoni

$$B_1 \ll \left(\frac{V(H^2)}{H^2} - \frac{V(1)}{H} + \int_H^{H^2} z^{-2} V(z) dz \right) \ln^2 NH, \quad (2.3.9)$$

ko'rinishida yozish mumkin. Bu yerda

$$V(X) = \sum_{h \leq X} \left| \sum_{n \leq N, n \neq f} e(hn^k \alpha) \right|. \quad (2.3.10)$$

(2.3.10) ga Koshi tengsizligi va $R = 2^{k-1}$, $W = k!(ND^{-1})^{k-1}$ deb olib

$$|V(X)|^R \leq X^{R-1} \sum_{h \leq X} \left| \sum_{n \leq N, n \equiv f} e(hn^k \alpha) \right|^R \ll X^{R-1} \times \\ \times \sum_{h \leq X} \left(\frac{X}{D} \right)^{R-k+\varepsilon} \left(\left(\frac{N}{D} \right)^{k-1} + \sum_{y \leq W} \min \left(\frac{N}{D}; \frac{1}{\|\alpha D^k y h\|} \right) \right)$$

ni hosil qilamiz.

$t = yh$ deb belgilash kiritamiz, u holda $t \leq k!(ND^{-1})^{k-1} X = Y$ va t ni

$t = yh$ ko‘rinishda ifodalashlar soni $\ll t^\varepsilon$ bo‘lgani uchun

$$|V(X)|^R \leq X^{R-1+\varepsilon} \left(\frac{N}{D} \right)^{R-k+\varepsilon} \left(\left(\frac{N}{D} \right)^{k-1} X + \sum_{y \leq W} \min \left(\frac{N}{D}; \frac{1}{\|\alpha D^k t\|} \right) \right),$$

deb yoza olamiz.

Bu ifodaning o‘ng tomoniga 2.3.1-lemmani qo‘llasak,

$$V(X) \ll (NXq)^\varepsilon \left(\frac{N}{D} \right) X \left(\frac{d^k}{q} + \left(\frac{N}{D} \right)^{-1} + \left(\frac{N}{D} \right)^{-k} \frac{q}{d^k X} \right)^{\frac{1}{R}} \quad (2.3.11)$$

baho kelib chiqadi.

(2.3.11) va $H > Nq\eta^{-1}$ ekanligidan

$$\frac{V(H^2)}{H^2} \ll (Nq\eta^{-1})^\varepsilon \frac{N}{D} \left(\frac{d^k}{q} + \frac{N}{D} + \left(\frac{D}{N} \right)^k \frac{q}{H^2 d^k} \right)^{\frac{1}{R}} \ll (Nq\eta^{-1})^\varepsilon \frac{N}{D} \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}},$$

va

$$\frac{V(1)}{H} \ll \frac{N}{DH} \ll \frac{N\eta}{DNq} \ll \eta(Dq)^{-1},$$

larga ega bo‘lamiz. Shuning uchun ham (2.3.9) dan

$$B_1 \ll (Nq\eta^{-1})^\varepsilon \frac{N}{D} \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}}, \quad (2.3.12)$$

bahoning o‘rinli ekanligi kelib chiqadi.

Shunday qilib, (2.3.7), (2.3.8) va (2.3.12) lardan

$$A_2 \ll (Nq\eta^{-1})^\varepsilon \frac{N}{D} \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}}. \quad (2.3.13)$$

Tenglik hosil bo'ladi.

Endi A_1 ni baholaymiz (2.3.3) dan

$$A_1 \ll \sum_{m \leq H} \min(m^{-1}, \eta) \left| \sum_{n \leq N, n \neq f} \Lambda(n) e(mn^k \alpha) \right|$$

foydalanib yozish mumkin. Bu yerdan esa Stiltes integralidan foydalanib

$$A_1 \ll \frac{G(H)}{H} - \eta G(1) - \int_{\eta^{-1}}^H u^{-2} G(u) du,$$

ga kelamiz. Bunda

$$G(X) = \sum_{m \leq X} \left| \sum_{n \leq N, n \neq f} \Lambda(n) e(mn^k \alpha) \right|.$$

Endi 2.3.1-teoremadan fiksirlangan $\varepsilon_0 > 0$ uchun

$$A_1 \ll ND^{-1}(\Delta(H) + \eta\Delta(1) + \Delta(\eta^{-1})\ln(H\eta)),$$

ekanligi kelib chiqadi. $\Delta(H)$ kamayuvchi funksiya va $\eta\Delta(1) \leq \Delta(\eta^{-1})$ bo'lgani uchun

$$A_1 \ll ND^{-1}\Delta(\eta^{-1})\ln(2Nq), \quad (2.3.14)$$

deb yoza olamiz.

(2.3.7), (2.3.13) va (2.3.14) lardan 2.3.1-lemma kelib chiqadi.

2.3.1-teoremaning isboti. $\delta = 1$ bo'lganda teorema trivial bo'lgani uchun

$\delta < 1$ deb faraz etamiz. Hozircha β va $\eta \left(0 \leq \eta < \frac{1}{2} \right)$ lar ixtiyoriy haqiqiy sonlar

edi. Endi $\beta = \gamma + \frac{\delta}{2}$ va $\eta = \frac{\delta}{2}$ deb olamiz. U holda 1-lemmadan

$$\begin{aligned} & \sum_{n \leq N, n \neq f} \Lambda(n) g_{\delta/2} \left(\alpha n^k - \gamma - \frac{\delta}{2} \right) = \delta \sum_{n \leq N, n \neq f} \Lambda(n) + \\ & + O \left(\frac{N}{D} \left(\Delta \left(\frac{2}{\delta} \right) \ln(2Nq) + (Nq\delta^{-1})^{\varepsilon_0} \left(\left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}} \right) \right) \right), \end{aligned} \quad (2.3.15)$$

ga ega bo‘lamiz. $g_n(x)$ funksiyaning aniqlanishiga ko‘ra ((2.3.1) ga qarang) agar

$$-\frac{\delta}{2} \leq \alpha n^k - \gamma - \frac{\delta}{2} - h < \frac{\delta}{2}, \text{ ya'ni } \gamma \leq \alpha n^k - h < \delta + \gamma \text{ shartni qanoatlantiruvchi h}$$

butun son mavjud bo‘lsa, u holda

$$g_{\delta/2} \left(\alpha n^k - \gamma - \frac{\delta}{2} \right) = 1.$$

Teorema shartiga ko‘ra $\gamma + \delta \leq 1$, shuning uchun ham bu yerdan $h = \lceil \alpha n^k \rceil$.

Demak, faqat va faqat $\gamma \leq \{ \alpha n^k \} < \delta + \gamma$ bajarilsagina

$$g_{\delta/2} \left(\alpha n^k - \gamma - \frac{\delta}{2} \right) = 1,$$

bo‘ladi. Buni e‘tiborga olib, (2.3.15) dan

$$\sum_{\substack{p \leq N, p \equiv f \\ \gamma \leq \{ \alpha n^k \} < \delta + \gamma}} \ln p - \delta \sum_{p \leq N, p \equiv f} \ln p \ll \frac{N}{D} \cdot \left(\Delta \left(\frac{2}{\delta} \right) \ln q + (Nq\delta^{-1})^{\varepsilon_0} \left(\frac{d^k}{q} + \frac{D}{N} \right)^{\frac{1}{R}} \right)$$

ni hosil qilamiz. Bundan esa isbotlanishi talab etilayotgan 2.3.1-teorema kelib chiqadi.

II bob uchun xulosa

Ikkinchi bobda Veyl teoremasi uning trigonometrik masalalarga tatbiqi, Veyl teotemasi tipidagi teoremlarning Varing masalasiga tatbiqi va ketma-ketliklar kasr qismlarining taqsimlanishini baholashda tatbiqlarini ko‘rib chiqildi. Bundan tashqari bu bobda Xua-Lo-Ken va Vinogradovlar lemmalari va ularni isbotlari keltirildi.

$$S_1 = \sum_{x=Q+1}^{Q+P} e^{2\pi i f(x)}, \quad f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \dots + \alpha_{n-1} x + \alpha_n \quad .$$

Bu yerda Q - butun son, P - musbat butun son, $\alpha, \alpha_1, \dots, \alpha_{n-1}, \alpha_n$ -ixtiyoriy haqiqiy sonlar, $n \geq 2$. Ushbu yig‘indi uchun biz birinchi navbatda bitta

tengsizlikni olamiz, uning yordamida biz yetakchi α koeffitsiyentining ratsional kasrga yaqinlashish xususiyatiga qarab yakuniy bahoni olindi.

$$|S|^{2^{n-1}} \leq P^{2^{n-1}-1} + P^{2^{n-1}-n-\varepsilon_1} \sum_{y=1}^{n!p^{n-1}} \min\left(P, \frac{1}{(\alpha, y)}\right)$$

Veyl yig'indi uchun yuqoridagi yakuniy baho olindi

III BOB. TUB SONLAR BO'YICHA OLINGAN TRIGONOMETRIK YIG'INDILARNING ADDITIV MASALALARNI YECHISHDA TATBIQLARI

3.1-§. Vinogradov teoremasi.

1937-yilda I.M. Vinogradov [18] tub sonlarga nisbatan yig'indilarni hisoblashning yangi usulini topdi, u eksponensial summaga murojat qildi.

$$f(a) = \sum_{p \leq n} e(\alpha p) \quad (3.1.1)$$

Bu yerda α haqiqiy, p esa a tub sonni boshlanishi va $e(\alpha) = \exp(2\pi i \alpha)$ ni bildiradi. $f(a)$ ga bo'lgan bahosidan foydalanib, Vinogradov Goldbaxning ternar muammosida Hardi va Littlvuds natijalarining yangi, so'zsiz isbotini keltirdi.

Uning natijasi Vinogradovning 3 ta asosiy teoremasi sifatida tanilgan.

3.1.1-teorema. (Vinogradov; 1937). Musbat n ni butun son uchun, $R(n)$ n ta sonni 3 ta tub son yig'indisi sifatida ifodalansin. Keyin

$$R(n) = \frac{n^2}{2(\log n)^3} \delta(n) + \theta\left(n^2 (\log n)^{-4}\right), \quad (3.1.2)$$

$$\delta(n) = \prod_{p/n} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p/n} \left(1 + \frac{1}{(p-1)^3}\right). \quad (3.1.3)$$

Xususan, har bir yetarlicha katta toq butun son uchta tub son yig'indisidan iborat.

Faraz qilaylik, M_i musbat butun sonlarning o'suvchi, cheksiz ketma-ketligi

$$a_{i1}, a_{i2}, \dots, a_{in} \dots, i = 1, 2, \dots, n \quad (1)$$

bo'lsin. Agar k ta M_1, M_2, \dots, M_k ketma-ketliklar (ularning orasida bir xillari ham bo'lishi mumkin) berilgan bo'lsa, elementlari

$$b_1 = a_{1j_1} + a_{2j_2} + \dots + a_{kj_k}, a_{1j_1} \in M_1, a_{2j_2} \in M_2, \dots, a_{kj_k} \in M_k \quad (2)$$

Tenglik bilan aniqlanuvchi b_1, b_2, \dots, b_n ketma-ketlikka berilgan ketma-ketliklarning yig'indisi deb ataladi va $M_1 + M_2 + \dots + M_k$ ko'rinishda belgilanadi.

$$P(M_i) = \inf_x \frac{N(x)}{x}.$$

Kattalikka M_i ketma-ketlikning zichligi deb ataladi. Bunda $N(x)$ (1)

ketma-ketlikagi x dan katta bo'lmagan hadlar sonini bildiradi.

Bu yerda quyidagi masalalarni hal etish muhim:

1. Agar M, M_1, M_2, \dots, M_k ketma-ketliklar berilgan bo'lsa, M dagi (2) shartni qanoatlantiruvchi elementlardan tuzilgan M' ketma-ketlikning zichligi qanday bo'ladi?

Shunga o'xshash M dagi (2) shartni qanoatlantirmaydigan elementlardan tuzilgan M'' ketma-ketlikning zichligini ham qarash mumkin.

2. b_j va k lar berilgan bo'lganda (2) shartni qanoatlantiruvchi $(a_{1j_1}, a_{2j_2}, \dots, a_{kj_k})$ lar sonini bildiruvchi $R_k(b_j)$ funksiyani tekshirish.

Boshqacha qilib aytganda, $R_k(b_j)$ funksiya uchun aniq yuqori va quyi chegaralarini 58rigono yoki asimtotik formula keltirib chiqarish.

Bu keltirilgan masalalar: Varing muammosi, Eylar-Goldbax muammosi, Xardi-Littlvud muammosi, Xua-Lo-Ken muammosi va boshqa ko'plab sonlar nazariyasining muammolarini o'z ichiga oladi.

Biz bu yerda yuqorida keltirilgan muammolardan ikkitasi:

Goldbax va Varing [19] muammolariga qisqacha to'xtalib o'tamiz.

1. 1742-yilda Xristian Goldbaxning Leonard Eyler [20] bilan yozishmalaridan kelib chiqqan Goldbax muammosini hozirgi zamon tilida quyidagicha ifodalash mumkin:

- har qanday 6 dan kichik bo'lmagan juft natural sonni ikkita toq tub son yig'indisi ko'rinishida ifodalash mumkin (Goldbaxning binar muammosi);

-har qanday 9 dan kichik bo'lmagan toq natural sonni 3 ta toq tub son yig'indisi ko'rinishida ifodalash mumkin (Goldbaxning ternar muammosi).

Bu muammolar birgalikda Eyler-goldbax muammosi deb ham yuritiladi.

Tushunarliki, Goldbaxning binar muammosining o'rinli ekanligidan ternar muammoning o'rinli ekanligi osonlik bilan kelib chiqadi. Haqiqatdan ham, agar $2n = p_1 + p_2$ bo'lsa, $2n + 3 = p_1 + p_2 + 3$, $n = 3, 4, \dots$ bajariladi.

Varing-Goldbax muammosiga kirishdan oldin, biz Varing muammosining eng muhim natijalarini taqdim etish uchun aylanma yo'lni amalga oshiramiz, chunki bu natijalar va Goldbax muammosi bo'yicha ishlar Varing-goldbax muammosiga asosiy turtki bo'lgan. Ehtimol qadimgi yunonlar har bir musbat sonni 4 ta tub sonning yig'indisi ekanligini 1-marta ko'rishgan ammo 1770-yilgacha Lagranj bu ajoyib dalilning to'liq isbotini topmagan.

Shuningdek, 1770-yilda Varing to'rt kvadratik teoremani umumlashtirishni taklif qildi, bu Varing muammosi deb nomlandi. Zamonaviy terminologiyada, Varning tahminlarida $k \geq 2$ sonida butun son borligi aytilgan $S = s(k)$ [21] shundayki, har bir natural son n , natural sonlarning $s(k)$ ko'p kuchlarining yig'indisidir. Ushbu taxminning bir nechta maxsus holatlari 19-asrda hal qilindi.

Varing-Goldbax muammosining bir necha variantlari va umumlashmalar mavjud bo'lib, ular yillar davomida katta e'tiborni tortdi. Masalan, diofont tenglamasi,

$$a_1 P_1^k + a_2 P_2^k + \dots + a_s P_s^k = n \quad (1)$$

deb hisoblash mumkin, bu yerda n, a_1, \dots, a_s , doimiy emas, balki butun sonlar. Ushbu shakl tenglamalari haqida bir necha savol berishimiz mumkin. Bundan tashqari (1) sistemani bilganimizda, yechimlarni $P_1, \dots, P_s \leq X$ bu yerda

X katta parameter. Ushbu turdagi taniqli muammo bu egizak prezumptsiya taxminidir; $p + 2$ ham tub bo‘lmagan p sonlari bor, ya‘ni $P_1 + P_2 = 2$ tenglamasi cheksiz ko‘p yechimga ega. Ushbu taxmin ikkilik Goldbax muammosi kabi juda qiyin va aslida 2 ta muammo juda ko‘p umumiy bo‘lgan. Xususan, egizaklarning taxminlari hanuzgacha ochiq bo‘lsada, Chenning Teorema 2 isboti juda ko‘p tub sonlar

$p + 2 = P_2$ ekanligini aniqlash uchun osongina o‘zgartirilishi mumkin.

$$f(P_1) + f(P_2) + \dots + f(P_s) = n.$$

Bu yerda $f(X) \in Z[X]$ yoki $x_1 + x_2 + \dots + x_n = n$ yoki (1) tipdagi tenglamalar tizimlari

$$P_1^j + P_2^j + \dots + P_s^j = n_j, \quad (1 \leq j \leq k),$$

tizim bilan bog‘liq. Ushbu tizimning eritmalari soni

$$R_{k,s}^*(n) \sim \frac{\Gamma^s\left(1 + \frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)} G_{k,s}^*(n) \frac{n^{\frac{s}{k}-1}}{(\log n)^s} \quad n \rightarrow \infty \quad da \quad (3.1.4)$$

ga o‘xshash assimptomatik formulani qondiradi, ammo bu assimtotik formuladagi asosiy termin (3.1.2) asosiy atamaga qaraganda kamroq tushiniladi.

Diofont tenglamalari tizimida vujudga keladigan yana bir klassik muammo tabiiy ravishda r tub sonlaridan trivial bo‘lmagan arifmetik progressiyalar mavjudligi bilan bog‘liq. $r \geq 3$ har bir butun sonida cheksiz ko‘p sonli arifmetik progressiyalar mavjudligi taxmin qilingan. Boshqacha qilib aytganda

$$P_i - 2P_{i+1} + P_{i+2} = 0 \quad (1 \leq i \leq r-2),$$

chiziqli tizimi cheksiz tublarda cheksiz ko‘p yechimga ega, $P_1, \dots, P_r \dots$

Agar $r=3$ bo‘lsa bu variantning 60rigono bilan o‘rnatilishi mumkin. Vinogradov uchta tub sonlar teoremasini isbotladi, ammo $r > 3$ bo‘lganida yuqoridagi tizim aylana usulidan mahrumdir.

Aslida, yaqin vaqtgacha uchta tub sonlar progressiyasini eng muhim tushuntirish quyidagi ikkita natijani keltiradi.

1. Heath-Brown [19] u yerda buni isbotlashga muvaffaq bo'ldi. Uchta tub son va P_r sonlarni cheksiz ko'p arifmetik proyeksiyalari mavjud.

2. Balog [20] har qanday r uchun P_1, \dots, P_r har xil tub son borligini, barcha o'rtacha ko'rsatkichlar $\frac{1}{2}(P_i + P_j)$ katta ekanligini isbotladi.

Shunday qilib, Green va Tao [21] o'z taxminlarini to'liq tasdiqlashlarini e'lon qilishganda, soha mutahassislari hayratda qoldilar. O'quvchi so'ngi bo'limda ularning g'oyalari va so'ngi ish bilan bog'liq qisqacha tavsifni topadi.

Va nihoyat (1.1) o'rniga $|x_1 + \dots + x_{s-a}| < \varepsilon$ o'rganish mumkin, bu yerda a haqiqiy son, ε kichik musbat son, x_1, \dots, x_s berilgan ketma-ketlikdan qiymatlarni oladigan haqiqiy o'zgaruvchilar.

Masalan, $x_j = p_j^e$ bu yerda $e > 1$ ni butun son emas deb belgilab, Waring-Goldbax masalasini tub sonlarning kasr kuchlariga umumlashtirishimiz mumkin.

Vinogradovning 3 ta asosiy teoremani isbotlashi, Hardi –Littlvud usulini keyinchalik tub sonlar bilan bog'liq bo'lgan qo'shimcha muammolarga qo'llash uchun asos bo'lib xizmat qiladi. 1-teorema nashr etilgandan ko'p o'tmay, Vinogradovning o'zi va Xua [22] hozirgi kunda Waring-Goldbach muammosi deb nomlanuvchi Waring muammosini asosiy o'zgaruvchilar bilan o'rganishni boshladilar. Ular barcha $k \geq 1$ uchun asimtotik formulasini (1.3) $k+h$ kuchlarga umumlashtirishga muvaffaq bo'lishdi va natijada ularning harakatlari quyidagi 3-teorama isbotiga olib keldi. Waring-Goldbach muammosi haqida mavjud bilimlarni tavsiflash uchun avvalo ba'zi bir nofatsiyalarni kiritish kerak. k musbat butun va p tub ko'paytiruvchisi bilan biz $p^\theta (k \text{ va } p^{\theta+1})k$ bo'lgan $\theta = \theta(k, P)$ butun (yagona) butun sonni aniqlaymiz va

$$\gamma = \gamma(k, p) \begin{cases} \theta + 2, \text{ agar } p = 2.2 / k \\ \theta + 1, \text{ aks holda } K(k) = \prod_{(p-1)/k} p^\gamma. \end{cases} \quad (3.1.5)$$

Xususan, bizda $k(1) = 2$. Agar butun sonning $3k$ kuchlarning yig'indisi $k+1$ dan kattaroq bo'lsa, $n \equiv s \pmod{k}$ tenglik shartini qanoqlantirishi kerakligini

ko`rsatish qiyin emas.

Bundan tashqari,

$$S(q, a) = \sum_{\substack{h=1 \\ (h, q)=1}}^q e\left(\frac{ah^k}{q}\right),$$

$$G_{k,s}^*(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a, q)=1}}^q \frac{S(q, a)^s}{\Phi(q)^s} e\left(\frac{-an}{q}\right). \quad (3.1.6)$$

Bu yerda (a, q) , a va q kattaliklarning eng katta bo`linuvchisi va $p(q)$ esa Eyler totent funksiya ya`ni $n \leq p$ ga nisbatan musbat butun sonlar soni. Keyingi natija 3.1.5 va 3.1.6 miqdorida belgilanadi.

3.1.2-teorema. k, s va n musbat butun sonlar bo`lsin va $R_{k,s}^*(n)$ diafont tenglamasining yechimlarini sonini

$$p_1^k + p_2^k + \dots + p_s^k = n, \quad (3.1.7)$$

deb belgilang.

p_1, p_2, \dots, p_n tub sonlar.

$$S \leq \begin{cases} 2^k + 1, & \text{agar } 1 \leq k \leq 5 \\ \frac{7}{8} \cdot 2^k + 1, & \text{agar } 6 \leq k \leq 8 \\ k^2 (\log k + \log \log k + O(1)), & \text{agar } k > 8 \end{cases}$$

Keyin

$$R_{k,s}^*(n) \ll \frac{\Gamma^s\left(1 + \frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)} G_{k,s}^*(n) \frac{n^{\frac{s}{k}-1}}{(\log n)^s} \text{ uchun } n \rightarrow \infty \quad ()$$

$G_{k,s}^*(n)$ (1.1.3) bilan belgilanadi. Bundan tashqari $G_{k,s}^*(n)$ qatorlar mutlaqo konvergentdir va agar $n \equiv s \pmod{k}$, keyin $G_{k,s}^*(n) \geq c_2(k, s) > 0$.

Xususan, bizda quyidagi tenglamalar mavjud. Muaffaqiyat 3.1. Har bir yetarlicha katta $n \equiv 5 \pmod{24}$ butun yig`indisi sifatida

ifodalanishi mumkin. Har bir yetarlicha katta toq butun sonni to'qqiz tub tubining yig'indisi sifatida ko'rsatish mumkin.

Xua Varing muammosinining $g(k)$ funksiyasiga o'xshash $H(k)$ funksiyasini taqdim etdi. $H(k)$ eng kichik butun sonlar sifatida aniqlanadi. (3.1.5) tenglamaning barcha tub p_1, \dots, p_n uchun $n \equiv s(\text{mod}(k, r))$ yechim topiladi. Barcha $k \geq 1$ uchun $H(k) = k + 1$ ekanligi taxmin qilinmoqda. Ammo k taxmin holi k ning biror bir qiymati uchun isbotlanmagan. Agar $k \leq 3$ bo'lsa, $H(k)$ uchun ma'lum bo'lgan yuqori chegaralar teorema 3 tomonidan berilgan, ya'ni $H(1) \leq 3, H(2) \leq 5, H(3) \leq 9, k \leq 4$ bo'lsa, adabiyotda eng yaxshi natijalar quyidagilar:

3.1.3-teorema. $k \geq 4$ butun son bo'lsin, va $H(k)$ yuqoridagi kabi bo'lsin .

Keyin

$$H(k) \leq \begin{cases} F(k), \text{ agar } 4 \leq k \leq 10, \\ k(4 \log k) + 2 \log \log k + O(1), \text{ agar } k > 10 \end{cases}$$

$F(k)$ quyidagi jadval bilan berilgan bo'lsa,

k	4	5	6	7	8	9	10
$F(k)$	14	21	33	46	63	83	107

$H(k)$ uchun chegara $4 \leq k \leq 10$,

3.1.3-teoremaning $k = 6$ va $8 < k \leq 10$ holatlari Thanigsalam [23] bilan bog'liq va $k = 4, 5, 7$ holatlar Kawada va Vuli [24] va Kumchev[25] ning so'ngi natijalari.

$k > 10$ bilan bog'langanlik –bu Xuaning qadimgi natijasidir, uning isbotini Xuan kitobida toppish mumkin.

3.2-§. Tub sonlar bo'yicha olingan birinchi darajali trigonometrik yig'indilarning baholari.

Ma'lumki, I. M. Vinogradov birinchi marta arifmetik progressiyaga tegishli tub sonlar buyicha olingan trigonometrik yig'indi

$$\sum_{\substack{p \leq N, p \equiv l \pmod{D} \\ (p, l) = 1}} e(ap) \quad (3.2.1)$$

moduli uchun trivial bo'lmagan baho olgan. Bo'laklab yig'ish usulidan foydalanib ko'rish qiyin emaski, (3.2.1) yig'indi moyiyati jixatidan

$$S_1(\alpha) = S_1(D, \alpha, N) = \sum_{\substack{n \leq N \\ m \equiv l \pmod{D}}} \Lambda(n) e(\alpha n), 1 \leq l \leq D, (l, D) = 1 \quad (3.2.2)$$

yig'indiga ekvivalent. A.F. Lavrik tomonida Drixli L funksiyasining nollari taqsimotining zichligi g'oyasi (3.2.2) ko'rinishdagi yig'indilarni baholashga tatbiq etildi. (3.2.2) yig'indining modulini baholashni

$$\sum_{x_q} \left| \psi(N, x_q) \right|, \quad (3.2.3)$$

yig'indini baholashda keltirish mumkin A.F. Lavrik ([26] qarang). Ishlarda (3.2.3) yig'indini Dirixle L -funksiyasi $L(s, \chi)$ nollarining zichligi to'g'risidagi ma'lumotlardan foydalanib baholangan. Vaughan R. C. esa (3.2.3) yig'indini A. F. Lavrikning taqribiy funksional tenglamasidan P.X.Gallagher [27] metodi yordamida keltirib chiqarilgan $L^2(s, \chi)$ uchun o'rta qiymat haqidagi teoremlardan foydanib tekshirgan.

Ushbu paragrafda A. F. Lavrik isbotining va Vaughan R. C. [2-teoremasi isboti g'oyalaridan foydalanib Dirixle L -funksiyasi $L(s, \chi)$ nollarining taqsimoti zichligi to'g'risidagi ma'lumotlarga tayanmaydigan ishdagi 1-teoremaga o'xshash teoremani isbotlaymiz. Bunda arifmetik progressiyaning ayirmasi darajali o'sish tartibiga ega bo'ladi.

Avvalo, (3.2.2) yig'indi haqidagi umumiy teoremani keltiramiz.

3.2.1-teorema. Agar γ, δ va $c \leq 1$ lar

$$\sum_{x_q} |\psi(N, x_q)| \ll (N + qN^{\frac{1}{2}} + q^\gamma N^\delta) L^c.$$

Bahoni qanoatlantiruvchi musbat sonlar bo'lsa, u holda

$$|\alpha - aq^{-1}| \leq 2N^{-1} \quad d = (D, q) \quad (a, q) = 1, \quad L = \ln N \quad \text{bo'lganda,}$$

(3.2.2) tenglik bilan aniqlanuvchi $S_1(\alpha)$ yig'indi uchun

baho o'rinli.

$$S_1(\alpha) \ll \left(\frac{d}{D} q^{\frac{1}{2}} N + q^{\frac{1}{2}} N^{\frac{1}{2}} + \left(\frac{d}{D} \right)^{1-\gamma} q^{\frac{1}{2}+\gamma} N^\delta \right) L^{c+1}.$$

Agar $L(s, \chi)$ funkshiyaning nollarining zichligi to'g'risidagi gipoteza o'rinli bo'lsa, (3.2.4) da $\gamma = 0, \delta = 1$ deb olish kerak.

Vauhgan R.C ning yig'indi uchun bahosidan foydalansak 1-teoremadan quyidagi (zichlik gipotezasiga tayanmaydigan) teorema kelib chiqadi.

3.2.2-teorema. Agar $|\alpha - aq^{-1}| \leq 2N^{-1} \quad (a, q) = 1, \quad d = (q, D)$ bo'lsa, u holda

$$S_1(\alpha) \ll \left(\frac{d}{D} q^{\frac{1}{2}} N + q^{\frac{1}{2}} N^{\frac{1}{2}} + \left(\frac{d}{D} \right)^{\frac{3}{8}} q^{\frac{1}{8}} N^{\frac{3}{4}} \right) L^{\frac{9}{2}},$$

bo'ladi.

Bu yerdan quyidagi natijalar kelib chiqadi:

3.2.1-natija. Agar $(D, q) = 1, q \ll L^c$, va $c \geq 12$ o'zgarimas son bo'lsa,

u holda ixtiyoriy $D \leq N^{\frac{2}{5}} L^{-c}$ uchun

$$S_1(\alpha) \ll \frac{N}{\varphi(D)} L^{-\frac{x+1}{2}},$$

baho o'rinli bo'ladi.

3.2.2-natija . Agar $P < q \leq NP^{-1}, 1 \leq P \leq N^{\frac{1}{3}}, |\alpha - aq^{-1}| \leq 2P(qN)^{-1}, (a, q) = 1$

bo'lsa u holda

$$S_1(\alpha) \ll \frac{N}{\varphi(D)} \Delta,$$

bajariladi.

$$\text{Bu yerda } \Delta = DP^{-\frac{1}{2}} L^{\frac{11}{2}}$$

3.3.3-natija. Agar $P < q \leq NP^{-1}$, $1 \leq P \leq N^{\frac{1}{3}}$, $|\alpha - aq^{-1}| \leq q^{-2}$, $(a, q) = 1$ va

$$\Delta = DP^{-\frac{1}{2}} L^{\frac{11}{2}} \text{ bo'lsa, u holda}$$

$$S_1(\alpha) \ll \frac{N}{\varphi(D)} \Delta$$

baho o'rinli bo'ladi.

3.2.2-teoremani A.F. Lavrik ishidagi 1-teoremaning natijasi

$$S_1(\alpha) \ll \left(\frac{d}{D} q^{\frac{1}{2}} N + q^{\frac{1}{2}} N^{\frac{1}{2}} + \left(\frac{d}{D} \right)^{\frac{2}{7}} q^{\frac{3}{4}} N^{\frac{5}{7}} \right) L^{18} \quad (3.2.5)$$

bilan taqqoslash ko'rsatadiki, agar ε bo'lganda 3.2.2 teoremaning natijasi barcha mavjud baholardan yaxshiroqdir. Yuqoridagi natijalar esa ilgari mavjud baholardan P ning o'zgarish diapazonining kengligi, $N^{\frac{1}{4}} \leq R \leq N^{\frac{1}{3}}$ bo'lganda olingan baho yaxshiligi, barcha baholarda logarifmik ko'paytuvchining darajasi kichikligi hamda isbotlash usuli bilan farq qiladi.

3.1.1-teoremaning isboti. Abelning bo'laklab yig'ish formulasidan foydalanib

$$S_1(\alpha) = S_1\left(\frac{a}{q}\right) e\left(N\left(\alpha - \frac{a}{q}\right)\right) - 2\pi i \left(\alpha - \frac{a}{q}\right) \int_0^N S_1\left(D, \frac{a}{q}, \xi\right) e\left(\xi\left(\alpha - \frac{a}{q}\right)\right) d\xi$$

deb yozib olamiz.

Bu yerdan

$$S_1(\alpha) = \left| S_1\left(\frac{a}{q}\right) \right| + \left| \alpha - \frac{a}{q} \right| \int_0^N \left| S_1\left(D, \frac{a}{q}, \xi\right) \right| d\xi. \quad (3.2.6)$$

Demak, $S_1(\alpha)$ yig'indi uchun trivial bo'lmagan baho olish uchun (3.2.2) yig'indi uchun $\alpha = \frac{a}{q}$, $(a, q) = 1$ ratsional nuqtalarda shunday baho olish yetarli ekan.

$$\sum_{\substack{n \leq N \\ (n, q) = 1}} \Lambda(n) e\left(\frac{an}{q}\right) \ll L^2$$

bo'lganining sababi $S_1\left(\frac{a}{q}\right)$ ni

$$S_1\left(\frac{a}{q}\right) = \sum_{\substack{n \leq N \\ (n, q) = 1}} \Lambda(n) e\left(\frac{an}{q}\right) + O(L^2) \quad (3.2.7)$$

ko'rinishida yoza olamiz. Ma'lumki $(n, q) = 1$ bo'lsa,

$$e\left(\frac{an}{q}\right) = \frac{1}{\varphi(q)} \sum_{x_q} \chi_q(an) \tau(\bar{\chi}_q)$$

bo'ladi, shuning uchun ham (3.2.7) dan

$$S_1\left(\frac{a}{q}\right) = \frac{1}{\varphi(q)} \sum_{x_q} \chi_q(an) \tau(\bar{\chi}_q) \sum_{\substack{n \leq N \\ n \in (\text{mod } D)}} \Lambda(n) \chi_q(n) + O(L^2) \quad (3.2.8)$$

kelib chiqadi.

(3.2.8) ning o'ng tomonidagi ohirgi yig'indini Dirixle xarakterlarining xossalaridan foydalanib

$$\frac{1}{\varphi(D)} \sum_{x_D} \bar{\chi}_D(e) \sum_{n \leq N} \Lambda(n) \chi_q(n) + \chi_D(n),$$

ko'rinishda yozish mumkin. Demak,

$$|\tau(\chi_q)| \leq q^{\frac{1}{2}}, \varphi(m) \gg \frac{m}{\dots m}, m > 1,$$

ekanligini inobatga olsak va

$$A = \sum_{x_q} \sum_{x_0} \left| \sum_{n \leq N} \Lambda(n) \chi_q(n) \chi_D(n) \right|$$

deb belgilash kiritsak, u holda $q \leq N$ ($q \leq N$ deb hisoblash mumkin, chunki agar $q > N$ bo'lsa, teorema trivial holda o'rinli bo'ladi) va $q \leq N$ lar uchun (3.2.8) dan

$$S\left(\frac{a}{q}\right) \ll \left(LD^{-1} q^{\frac{1}{2}} A + L^2 \right), \quad (3.2.9)$$

kelib chiqadi.

Xarakterlar ko'paytmasi yetakchi moduli ko'paytuvchilar yetakchi modullarining eng kichik umumiy karralisining bo'luvchisiga teng bo'lgan xarakter bo'lgani uchun $d = (q, D)$ deb olib, $\chi_q(n)\chi_D(n) = \chi_k(n)$ ga ega bo'lamiz. Bunda $k = qDd^{-1}$. Endi A da xarakterlar bo'yicha ikki karrali yig'indidan oddiy yig'indiga o'tib

$$A \leq d \sum_{\chi_i} |\psi(N, \chi_k)|, \quad (3.2.10)$$

ni hosil qilamiz. Bu yerda

$$\psi(N, \chi_k) = \sum_{n \leq N} \Lambda(n) \chi(n).$$

Shunday qilib, (3.2.9) va (3.2.10) lardan

$$S_1\left(\frac{a}{q}\right) \ll \left(\frac{dL}{D} q^{\frac{1}{2}} \sum_{\chi_i} |\psi(N, \chi_k)| + L^2 \right), \quad (3.2.11)$$

ga ega bo'lamiz. Bu yerdan (3.2.4) bahoga asosan

$$\begin{aligned} S_1\left(\frac{a}{q}\right) &\ll \frac{dL}{Dq^{\frac{1}{2}}} \left(N + kN^{\frac{1}{2}} + k^\gamma N^\delta \right) L^c + L^2 \ll \\ &\ll \left(\frac{d}{D} q^{\frac{1}{2}} N + q^{\frac{1}{2}} N^{\frac{1}{2}} + \left(\frac{d}{D} \right)^{1-\gamma} q^{\frac{1}{2}+\gamma} N \right) \end{aligned} \quad (3.2.12)$$

bo'ladi.

Endi agar $|\alpha - aq^{-1}| < 2N^{-1}$ ekanligini e'tiborga olsak, (3.2.6) va (3.2.12)

lardan 3.2.1- teorema kelib chiqadi.

3.2.2-teoremaning isboti. Vaughan R.C 2-teoremasiga ko'ra

$$\sum_{x_1} \max_{y, \chi_k} \psi(y, \chi_k) \ll NL^3 + N^{\frac{3}{4}} k^{\frac{5}{4}} L^{\frac{23}{8}} + N^{\frac{1}{2}} k L^{\frac{7}{2}}.$$

Bu bahodan foydalanib (1.11) dan

$$S_1\left(\frac{a}{q}\right) \ll \frac{d}{D} N q^{-\frac{1}{2}} L^4 + \left(\frac{d}{D}\right)^{\frac{3}{8}} N^{\frac{3}{4}} q^{\frac{1}{8}} L^{\frac{3}{8}} + N^{\frac{1}{2}} q^{\frac{1}{2}} L^{\frac{9}{2}}, \quad (3.2.13)$$

ga ega bo'lamiz.

Endi (3.2.6) va (3.2.13) dan $|\alpha - aq^{-1}| \leq 2N^{-1}$ bo'lganda 3.3.2 teorema kelib chiqadi.

3.2.1 va 3.2.2-natijalar ko'rsatilgan shartlarda 3.2.2-teoremadan to'g'ridan-to'g'ri kelib chiqadi. 3.2.1-natijadagi D ni $D \leq N^{\frac{2}{5}} L^{-c}$ shartni qanoatlantiruvchi qilib tanlashi (3.2.13) dagi

$$\left(\frac{d}{D}\right)^{\frac{3}{4}} q^{\frac{1}{8}} L^{\frac{31}{8}},$$

x had bilan bog'liq.

3.2.3-natijaning isboti esa G. Montgomeridagi 16.3 natijaning isbotiga o'xshash bajariladi.

Ta'kidlash kerakki, A.Balog, A. Perellilar Von ayniyati (A. A. Karatsubaning III-bobdagi 9-masalasi) dan foydalanib, elementar usul bilan

$$S_1(\alpha) \ll \left(\frac{dN}{Dq^{\frac{1}{2}}} + \frac{q^{\frac{1}{2}} N^{\frac{1}{2}}}{d^{\frac{1}{2}}} + \frac{N^{\frac{4}{5}}}{D^{\frac{2}{5}}} \right) L^3,$$

ekanligini isbotladilar. Lekin $d = (D, q) = 1$ va $q \leq N^{\frac{2}{5}} D^{\frac{1}{5}} L^{-12}$ bo'lganda 3.2.2-teorema bu bahodan yaxshi.

3.3-§. Tub sonlar bo'yicha olingan birinchi darajali trigonometrik yig'indilarning additiv masalalarni yechishdagi tatbiqlari

Ma'lumki, sonlarning analitik nazariyasidagi trigonometrik masalalarda

$$S_k(D, \alpha, N) = \sum_{\substack{n \leq N \\ n \equiv l \pmod{D}}} \Lambda(n) e(\alpha n^k),$$

ko'rinishdagi trigonometrik yig'indilarning trivial bo'lmagan baholari muhim ahamiyatga ega. Bunday baholarni olishga ko'plab ishlar bag'ishlangan.

A .Ghosh $S_2(1, h\alpha, N) = S_2(h\alpha)$ uchun yangi baho olgan va

$$\sum_{h \leq H} S_2(h\alpha) \ll HN^{1+\varepsilon} \left(q^{-1} + N^{-\frac{1}{2}} + qH^{-1}N^{-2} \right)^{\frac{1}{4}}, \quad (3.3.1)$$

ning bajarilishini ko'rsatgan. Biz (3.3.1)-bahoni arifmetik progressiya uchun umumlashtiramiz (I. Allakov [28]). Ushbu paragrafning asosiy maqsadi quyidagi teoremani isbotlashdan iborat.

3.3.1-teorema. Agar $|\alpha - aq^{-1}| \leq q^{-2}$, va $D^2 \leq N$ bo'lsa, u holda ixtiyoriy soni uchun

$$\sum_{h < H} |S_2(D, h\alpha, N)| \ll \frac{HN^{1+\varepsilon}}{D} \left(\frac{d^2}{q} + \frac{D}{N^{\frac{1}{2}}} + \frac{qD}{HN^2 d} \right)^{\frac{1}{4}}, \quad (3.3.2)$$

baho o'rinli. Bu yerda $d = (q, D)$ va Vinogradov simvolidagi doimiy faqat ε ga bog'liq.

Agar zarurat bo'lsa, (3.3.2) ning o'ng tomonidagi $\varepsilon > 0$ ni $\frac{c}{\ln \ln N}$ bilan almashtirish mumkin.

Bu teoremadan quyidagi natijalar kelib chiqadi:

3.3.1-natija. Ushbu

$$S_2(D, \alpha, N) \ll D^{-1} N^{1+\varepsilon} \left(d^2 q^{-1} + DN^{-\frac{1}{2}} + qDN^{-2} d^{-1} \right)^{\frac{1}{4}} \quad (3.3.3)$$

baho o‘rinli

3.3.2-natija. Agar $D \leq N^\varepsilon$ va $N^{\frac{1}{2}} < q \leq N^{\frac{3}{2}}$ bo‘lsa

$$S_2(D, \alpha, N) \ll D^{-1} N^{\frac{7}{8} + \varepsilon},$$

bo‘ladi .

1-natija. Agar A. Ghoshning 2-teoremasining arifmetik progressiya uchun umumlashmasidir. (1.16) ko‘rinishdagi baholar odatda q ning o‘zgarish diapazoni katta bo‘lganda keng qo‘llaniladi. Shu nuqtai nazardan (3.3.3)- baho muhimdir.

3.3.3-teoremani bevosita isbotlashdan oldin uni isbotlash uchun zarur bo‘lgan lemmalarni keltiramiz. Bu lemmalardan dastlabki ikkitasi I. M. Vinogradovga tegishli bo‘lib, ko‘pchilik mutaxassislariga yaxshi tanish. Bundan keyin $m \equiv f \pmod{D}$ ning o‘rniga q yozuvdan foydalanamiz.

3.3.1-lemma. Agar $\alpha, (0 < \alpha < 1)$ haqiqiy son va X, X' lar musbat butun sonlar bo‘lsa, u holda

$$\sum_{\substack{x < m \leq x' \\ m \equiv f}} e(m\alpha) \ll e\left(\frac{X'}{D} + 1; \|\alpha D\|^{-1}\right),$$

baho o‘rinli.

3.3.2-lemma. Agar X va Y lar musbat sonlar , α esa o‘zaro tub α va q butun sonlari uchun $|\alpha - aq^{-1}| \leq q^{-2}$ shartlarni qanoatlantirsa, u holda

$$\sum_{m \leq X} \min(Y; \|\alpha m\|^{-1}) \ll XYq^{-1} + (X + q) \ln 2q$$

$$\sum_{m \leq X} \min\left(\frac{XY}{m}; \|\alpha m\|^{-1}\right) \ll (XYq^{-1} + X + q) \ln(2XYq)$$

Munosabat o‘rinli. Endi X, X', Y, Y' lar X, X', Y, Y' va $XY \leq N$ shartlarni qanoatlantiruvchi butun sonlar bo‘lsin

$$S_1 = \sum_{n \leq H} \left| \sum_m \omega(m) \sum_{(m,n) \in G} \psi(n) e(hm^2 n^2 \alpha) \right| \quad (3.3.4)$$

deb belgilab olamiz. Bu yerda

$$G = \left\{ (m, n) \mid X < m \leq X', Y < n \leq Y', m \equiv f_1, n \equiv f_2, m * n \leq N \right\}, \omega(m) \quad \text{va} \quad \psi(n)$$

lar

$$|\omega(m)| \leq \tau(m) \ln m, \quad |\psi(n)| \leq \tau(n) \ln n$$

(3.3.5)

shartni qanoatlantiruvchi arifmetik funksiyalar. Tushunarliki,

3.3.3-lemma. Agar $\ln q \ll \ln N$ bo'lsa, u holda (3.3.4) tenglik bilan aniqlanuvchi S_1 yig'indi uchun quyidagi baholar o'rinli:

à) Agar barcha $n \in (Y, Y']$ lar uchun $\psi(n) = 1$ bo'lsa, u holda

$$S_1 \ll (NH)^\varepsilon \left(\frac{N^{\frac{1}{2}} H M_2}{D^{\frac{1}{2}}} + \frac{d N H M^{\frac{1}{2}}}{D q^{\frac{1}{2}}} + \left(\frac{H M_2 q}{d D} \right)^{\frac{1}{2}} \right) \quad (3.3.6)$$

b) Agar biror $n \in (Y, Y']$ uchun $\psi(n) \neq 1$ bo'lsa, u holda

$$S_1 \ll (NH)^k \left(\frac{N^{\frac{1}{2}} H X^{\frac{1}{2}}}{D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}} N H}{D q^{\frac{1}{4}}} + \frac{H N^{\frac{3}{4}} Y^{\frac{1}{4}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{d^{\frac{1}{4}} D^{\frac{3}{4}}} \right) \quad (3.3.7)$$

$$S_2 = \sum_{h \in H} \left| \sum_{\substack{M_1 \in m \in M_2 \\ m=K_1}} \omega(m) \sum_{\substack{N_1 \leq N_2 \\ m=\frac{1}{2}}} \psi(n) e(h m^2 n^2 \alpha) \right|$$

yig'indini baholashga tatbiq etamiz. Bu yerda M_1, M_2, N_1, N_2 —lar $M_1 < M_2, N_1 < N_2$ va $M_1 N_1 \leq N$ shartlarni qanoatlantiruvchi musbat butun sonlar, qolgan barcha parametrlar esa 3.3.3-lemmadagi singari aniqlanadi. S_2 ni baholash uchun $(M_1, M_2]$ va $(N_1, N_2]$ intervallardan har birini $(X, X']$ va $(Y, Y']$ ko'rinishdagi $\ln q \ll \ln N$ ta bo'lakchalarga (qism intervallarga) bo'lamiz. Bunda $M_1 \leq X < X' \leq 2X \leq M_2$ va $N_1 \leq Y < Y' \leq 2Y \leq N_2$.

Agar $XY > N$ bo'lsa G to'plam bo'sh bo'lgani uchun $XY < N$ deb hisoblashimiz mumkin. Bulardan foydalanib, S_2 ni $S_1 = S_1(X, Y)$ yig'indi orqali

$$S_2 \leq \sum_X \sum_Y S_1(X, Y) \ll (\max S_1(X, Y)) \ln^2 N,$$

ko'rinishda ifodalash mumkin. Bu yerdan quyidagi tasdiq kelib chiqadi.

3.3.4-lemma. A) Agar barcha $n \in (N_1; N_2]$ lar uchun $\psi(n) = 1$ bajarilsa u holda

$$S_2 \ll (NH)^\varepsilon \left(\frac{N^{\frac{1}{2}} HM_2}{D^2} + \frac{dNH M_2^{\frac{1}{2}}}{Dq^{\frac{1}{2}}} + \left(\frac{HM_2 q}{dD} \right)^{\frac{1}{2}} \right),$$

bo'ladi.

b). Agar biror $n \in (Y, Y']$ uchun $\psi(n) \neq 1$ bajarilsa, u holda

$$S_2 \ll (NH)^c \left(\frac{N^{\frac{1}{2}} HM_2^{\frac{1}{2}}}{D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}} NH}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{3}{4}} N_2^{\frac{1}{4}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{D^{\frac{3}{4}} d^{\frac{1}{4}}} \right)$$

baho o'rinli bo'ladi.

Isboti. S yig'indiga Koshi tentsizligini [29] qo'llasak,

$$S_1 \leq \left[\sum_1 (\sum_2 + \sum_3) \right]^{\frac{1}{2}}, \quad (3.3.8)$$

hosil bo'ladi. Bunda

$$\begin{aligned} \sum_1 &= \sum_{h \leq H} \sum_{\substack{x \leq m \leq X' \\ m = f_1}} |\omega(m)|^2, \quad \sum_1 = \sum_{h \leq H} \sum_m \sum_{\substack{(m,n) \in G \\ n \leq H}} |\omega(m)|^2, \\ \sum_{(m,n) \in G, i=1,2} &= \sum_{m_1 \neq m_2} \psi(n_1) \psi(n_2) e(hm^2(n_1^2 - n_2^2)\alpha) \end{aligned}$$

(3.3.5) shartni va

$$\tau_k^\varepsilon(n) \ll n^\varepsilon,$$

bahoni inobatga olib

$$\sum_1 \ll X^{1+\varepsilon} HD^{-1}, \quad \sum_2 \ll NY^\varepsilon HD^{-2}, \quad (3.3.10)$$

larga ega bo'lamiz.

Endi \sum_3 ni baholaymiz. Ikki holni qaraymiz.

À). Barcha $n \in (Y, Y']$ lar uchun $\psi(n) = 1$ bo'lsin. U holda \sum_3 da $n_1 - n_2 = y$ deb olsak.

$$\sum_3 \ll \sum_{h \leq H} \sum_{x \in m \leq x} \sum_{r \leq y \leq r'} \left| \sum_{\substack{n=0 \\ (m,n) \in G \\ (m,n+y) \in G}} e(2hm^2 y n \alpha) \right|,$$

bo'ladi. $s = 2hm^2 y$ deb belgilash kiritamiz, u holda $2DX^2 \ll |s| \leq 16NHX$ va s ni bunday ko'rinishda ifodalashlar soni $\tau_3(|s|)$ dan ko'p emas. Shuning uchun ham

$$\sum_3 < \sum_{\substack{|s| < I < NHX \\ s=0}} \tau_3(|s|) \left| \sum_{\substack{Y \in n < r \\ n=f}} e(sn \alpha) \right|, \quad (3.3.11)$$

Bunda 3.3.1-lemma va (3.3.9) bahodan foydalansak,

$$\sum_3 \ll (NHX)^\varepsilon \sum_{|r| < I < N = NHX} \min\left(\frac{Y'}{D}; \|sD\alpha\|^{-1}\right),$$

kelib chiqadi.

$$Y' \leq \frac{N}{X'} = \frac{N}{|s|} |s| \frac{1}{X'} < \frac{16N^2 H}{|s|} \text{ va } d = (q, D), d \leq (D^2, q) < d^2,$$

bo'lganligi sababli (3.3.11) ning o'ng tomoniga 3.3.2-lemmani qo'llab,

$$\sum_3 (NHX)^c \left(\frac{d^2 N^2 H}{Dq} + NHX + \frac{q}{d} \right) \ln \left(\frac{32N^2 Hq}{dD} \right), \quad (3.3.12)$$

ni hosil qilamiz.

(3.3.10), (3.3.12) va $\ln q \ll \ln N$ larni e'tiborga olsak (3.3.8) dan teoremaning

a) tasdig'i kelib chiqadi.

b). Endi biror $n \in (Y, Y']$ uchun $\psi(n) \neq 1$ bajarilsin. Bu holda \sum_3 ni

$$\sum_3 = \sum_{h \leq H} \sum_{m \leq m \leq x'} \sum_{(m,t) \in \Phi} Z(t) e(hm^2 t \alpha)$$

ko'rinishda yozib olamiz. Bunda

$$\Phi = \left\{ (m, t) \mid X < m \leq X', D \leq |t| < Y'^2, |m^2 t| \leq N^2, m \equiv f, t \equiv 0 \right\},$$

va

$$Z(t) = \sum_{\substack{m=m_2, n_1^2 - n_2^2 - 1 \\ (m, n_1) \in G, i=1, 2, \dots}} \psi(n_1)\psi(n_2).$$

Tushunarliki,

$$|\Phi| \ll \frac{NY}{D^2}, \sum_{h \leq H} \sum_{\substack{D \leq |t| < Y^2 \\ t=0}} |Z(t)|^2 \ll \frac{HY^{2+\varepsilon}}{D}$$

Σ_3 da yig'ish tartibini o'zgartirib Koshi tengsizligini qo'llasak,

$$\begin{aligned} \Sigma_3 &= \sum_{h \leq H} \sum_{D \leq |t| < Y^2} Z(t) \sum_{(m, n) \in \Phi} e(htm^2\alpha) \leq \\ &\leq \left(\sum_{h \leq H} \sum_{D \leq |t| < Y^2} |Z(t)|^2 \right)^{\frac{1}{2}} \left(\sum_{h \leq H} \left(\sum_{D \leq |t| < Y^2} \left| \sum_{(m, n) \in \Phi} e(htm^2\alpha) \right|^2 \right) \right)^{\frac{1}{2}} \ll \\ &\ll \left(\frac{HY^{2+\varepsilon}}{D} \right)^{\frac{1}{2}} \left(HYND^{-2} + \Sigma_4 \right)^{\frac{1}{2}}, \end{aligned}$$

hosil bo'ladi. Bunda

$$\Sigma_4 = \sum_{h \leq H} \sum_{D \leq |t| < Y^2} \sum_{\substack{m_1 \neq m_2 \\ (m_1, t) \in \Phi, i=1, 2}} e(ht(m_1^2 - m_2^2)\alpha).$$

Bu Σ_4 yig'indini a) holdagi Σ_3 yig'indi singari baholab

$$\begin{aligned} \Sigma_4 &\ll \sum_{|s| < 6NHX} \tau_3(|s|) |e| \left(\frac{N}{D|t|^{1/2}}; |s \setminus D\alpha|^{-1} \right) \ll \\ &\ll (NH)^s \left(\frac{d^2 N^2 H}{qD} + NHY + \frac{q}{d} \right) \ln \left(\frac{2N^2 Hq}{dD} \right), \end{aligned}$$

bahoga ega bo'lamiz.

(3.3.14) ga ko'ra (3.3.13) dan

$$\Sigma_3 \ll (NH)^\varepsilon \left(\frac{dNHY}{Dq^{1/2}} + \frac{N^{\frac{1}{2}} Y^{\frac{1}{2}} H}{D^{\frac{1}{2}}} + \frac{H^{\frac{1}{2}} q^{\frac{1}{2}} Y}{D^{\frac{1}{2}} d^{\frac{1}{2}}} \right),$$

kelib chiqadi. Endi (3.3.8), (3.3.10) va (3.3.15) lardan teoremaning á) tasdig'i kelib chiqadi.

Endi 3.3.3-lemmada S_1 yig'indi uchun olingan baholarni

$$S_2 = \sum_{h \in H} \left| \sum_{\substack{M_1 \in m \in M_2 \\ m=K_1}} \omega(m) \sum_{\substack{N_1 \in n \in N_2 \\ m=2}} \psi(n) e(hm^2 n^2 \alpha) \right|,$$

yig'indini baholashga tatbiq etamiz. Bu yerda M_1, M_2, N_1, N_2 –lar $M_1 < M_2, N_1 < N_2$ va $M_1 N_1 \leq N$ shartlarni qanoatlantiruvchi musbat butun sonlar, qolgan barcha parametrlar esa 3.3.3-lemmadagi singari aniqlanadi. S_2 ni baholash uchun $(M_1, M_2]$ va $(N_1, N_2]$ intervallardan har birini $(X, X']$ va $(Y, Y']$ ko'rinishdagi $\ln q \ll \ln N$ ta bo'lakchalarga (qism intervallarga) bo'lamiz. Bunda $M_1 \leq X < X' \leq 2X \leq M_2$ va $N_1 \leq Y < Y' \leq 2Y \leq N_2$

Agar $XY > N$ bo'lsa G to'plam bo'sh bo'lgani uchun $XY < N$ deb hisoblashimiz mumkin. Bulardan foydalanib, S_2 ni $S_1 = S_1(X, Y)$ yig'indi orqali

$$S_2 \leq \sum_x \sum_y S_1(XY) \ll (\max S_1(X, Y)) \ln^2 N,$$

ko'rinishda ifodalash mumkin. Bu yerdan quyidagi tasdiq kelib chiqadi.

3.3.4-lemma. A). Agar barcha $n \in (N_1, N_2]$ lar uchun $\psi(n) = 1$ bajarilsa, u holda

$$S_2 \ll (NH)^\epsilon \left(\frac{N^{1/2} H M_2}{D^{1/2}} + \frac{d N H M_2^{1/2}}{D q^{1/2}} + \left(\frac{H M_2 q}{d D} \right)^{1/2} \right),$$

bo'ladi.

b). Agar biror $n \in (Y, Y']$ uchun $\psi(n) \neq 1$ bajarilsa, u holda

$$S_2 \ll (NH)^c \left(\frac{N^{1/2} H M_2^{1/2}}{D^{3/2}} + \frac{d^{1/2} H N}{D q^{1/4}} + \frac{H N^{3/4} N_2^{1/4}}{D^{3/4}} + \frac{H^{3/4} q^{1/4} N^{1/2}}{D^{3/4} d^{1/4}} \right),$$

baho o'rinli bo'ladi.

3.3.1- teoremaning isboti. $q > N^2 H d D^{-1}$ bo'lsin. U holda (3.3.2) bahoning o'ng tomonini $\ll H N^{1+\epsilon} D^{-1}$ ko'rinishda ifodalash mumkin va (3.3.2) baho trivial bajariladi.

Endi $H > N$, $q \leq N^2 H d D^{-1}$ bo'lsin. U holda Koshi tengsizligiga[30] asosan

$$\left(\sum_{h=1}^H S^2(D, h\alpha, N) \leq H^{1/2} \left(\sum_{h=1}^H \left| \sum_{\substack{1 \leq n_1 \leq N \\ n_2 = n_1}} \Lambda(n_1) \Lambda(n_2) e(hn_1^2 \alpha) \right| \right)^{1/2} \right) \ll \\ \ll H^{1/2} \left(\frac{HN}{D} \ln N + \sum_{\substack{n \in n_1, m=n \\ \leq \Delta, m_2 \leq N}} \Lambda(n_1) \Lambda(n_2) e(h(n_1^2 - n_2^2) \alpha) \right)^{1/2},$$

ga ega bo‘lamiz. Endi

$$\sum_4 Z_1(l) = \sum_{\substack{1 \leq n_1, m_1 \leq N \\ m_1^2 - n_1^2 = \frac{l}{m_1} \neq n_2}} \Lambda(n_1) \Lambda(n_2) \leq (\ln N)^2 \tau(|l|) \sum_4$$

deb belgilash kiritamiz. U holda

$$\sum_{h=1}^H S_2(D, h\alpha, N) \ll \left(D^{-1} N \ln N \right)^{\frac{1}{2}} \cdot H + H^{\frac{1}{2}} \left| \sum_{h \leq H} \sum_{\substack{k < N^2 \\ l=0}} Z_1(t) e(hl\alpha) \right|^{\frac{1}{2}},$$

bo‘ladi. Bunda yig‘ish tartibini o‘zgartirib, 3.3.1 va 3.3.2 lemmalardan foydalansak,

$$\sum_{h=1}^H S_2(D, h\alpha, N) \ll \frac{HN^{\frac{1}{2}}}{D^{\frac{1}{2}}} \ln^{\frac{1}{2}} N + H^{\frac{1}{2}} N^\varepsilon \left| \sum_{h=H}^* \sum_{l=0} e(hl\alpha) \right|^{\frac{1}{2}} \ll \\ \ll \frac{HN^{\frac{1}{2}}}{D^{\frac{1}{2}}} \ln^{\frac{1}{2}} N + H^{\frac{1}{2}} N^\varepsilon \left(\sum_{h < H} e \left(\frac{N^2}{D}; \|hD\alpha\|^{-1} \left| \frac{1}{2} \right| \right) \right) \ll \\ \ll H \left(\frac{N \ln N}{D} \right)^{\frac{1}{2}} + H^{\frac{1}{2}} N^\varepsilon \left(\frac{dN^2 H}{qD} + H + \frac{q}{D} \right)^{\frac{1}{2}} \ln^{\frac{1}{2}} N \ll \\ \ll \frac{HN^{1+\varepsilon}}{D} \left(\frac{D^{\frac{1}{2}}}{N^{\frac{1}{2}}} + \frac{Dq^{\frac{1}{4}}}{H^{\frac{1}{4}} N^{\frac{3}{4}} d^{\frac{1}{2}}} + \frac{D^{\frac{1}{2}} d^{\frac{1}{2}}}{q^{\frac{1}{4}}} \right) \ll \frac{HN^{1+\varepsilon}}{D} \left(\frac{D^2}{N^2} + \frac{D^4 q}{HN^3 d^2} + \frac{D^2 d^2}{q} \right)^{\frac{1}{4}},$$

kelib chiqadi. Bundan esa (3.3.2) bahoga ega bo‘lamiz.

Endi $H < N$, $q \leq N^2 H d D^{-1}$ bo‘lsin. Ushbu

$$\sum_{u < n \leq N} F(1, n) + \sum_{u \in n \leq Nn^{-1}} \sum_{u < m \leq Nn^{-1}} t_m F(m, n) = \sum_{s \leq W} \mu(s) \sum_{W < n \leq Ns^{-1}} \sum_{r \leq N(s)^{-1}} F(sr, n) \quad (3.3.16)$$

ayniyatni qaraymiz (A. A. Karatsuba ning III-bobidagi 9-masala[31]). Bunda $F(m, n)$ - natural m va n argumentli ixtiyoriy funksiya, $\mu(s)$ esa Myobus funksiyasi, ($1 \leq u \leq N$) ixtiyoriy butun son,

$$t_m = \sum_{s \leq u, s/m} \mu(s)$$

$F(m, n)$ funksiyani quyidagicha aniqlaymiz. Agar $u < n < Nm^{-1}$ va $m \equiv f_1$, $n \equiv f_2$ bo'lsa, $F(m, n) = \Lambda(n)e(hm^2n^2\alpha)$, qolgan hollarda esa $F(m, n) = 0$ deb olamiz hamda $u < N^{1/3}$ shart bajariladi deb qaraymiz. U holda (1.2.9) ayniyatdan

$$S_2(D, h\alpha, N) = L_1 - L_2 - L_3 + O(N^{\frac{1}{3}}D^{-1}), \quad (3.2.17)$$

ga ega bo'lamiz. Bu yerda

$$L_1(h) = \int_1^N L_1(\beta) \beta^{-1} d\beta,$$

$$L_1(\beta) = \sum_{s=f_1+r} \mu(s) \sum_{\substack{\beta < k < Nr^{-1} \\ k=\gamma_2}} e(hs^2k^2\alpha), (\bar{r} \cdot r \equiv 1);$$

$$L_2(h) = \sum_{\substack{l \leq w^2 \\ l=f}} C_l \sum_{r \leq N||^{-1}} e(hl^2r^2\alpha), (\bar{s} \cdot s \equiv 1; f_3 \equiv f_1\bar{r}f_2),$$

$$C_l = \sum_{\substack{s, l=su \\ s=f_1}} \mu(s) \Lambda(n) \leq \tau(l)el$$

$$L_3(h) = \sum_{\substack{u \in m \leq Nn^{-1} \\ m=f_1}} t_m \sum_{\substack{w < n \leq Nm^{-1} \\ n=\gamma_2}} \Lambda(n)e(hm^2n^2\alpha)$$

(3.2.17) dan

$$\sum_{h \leq H} S_2(D, h\alpha, N) = L_1 + L_2 + L_3 + O(N^{\frac{1}{3}}D^{-1}) \quad (3.2.18)$$

ekanligi kelib chiqadi. Bu yerda

$$L_i = \sum_{h \leq H} L_i(h), \quad i = 1, 2, 3.$$

L_1 ni baholaymiz. 3.3.4-lemmaning a) qismida $L_1 = 1$,

$M_2 = \min(u, N\beta^{-1}) \leq u$, $N_1 = \beta$, $N_2 = N$ va $|\psi(s)| = |\mu(s)| \leq 1$ deb olamiz, u holda

$$L_2 \ll N^c \left(\frac{HN^{\frac{1}{2}}u}{D^{\frac{1}{2}}} + \frac{dHNu^{\frac{1}{2}}}{Dq^{\frac{1}{2}}} + \left(\frac{Hqu}{dD} \right)^{\frac{1}{2}} \right),$$

bahoni hosil qilamiz.

L_2 ni baholash uchun $L_1(h)$ yig'indini quyidagicha qism yig'indilarga ajratamiz

$$+ \sum_{\substack{u < m \leq N^{\frac{1}{2}} \\ m = N_1}} \sum_{\substack{N^{\frac{1}{2}} r \leq N^{-1} \\ m}} C_m e(hm^2 r^2 \alpha) = \sum_{j=1}^5 L_2^{(j)}(h),$$

$$\text{U holda } L_2 \leq \sum_{h \leq H} \sum_{j=1}^5 L_2^{(j)}(h) = \sum_{j=1}^5 L_2^{(j)}, \text{ bo'ladi.}$$

endi har bir $L_2^{(\rho)}$ yig'indini 3.3.4 lemmadan foydalanib baholaymiz.

$L_2^{(1)}$ ni baholash uchun 3.3.4-lemmaning a) tasdig'ida $M_1 = 1$, $M_2 = u$,

$N_1 = 1$, $N_2 = N$, $\omega(m) = C_m$ deb olamiz va bu holda

$$L_2^{(1)} \ll N^\varepsilon \left(\frac{HN^{\frac{1}{2}}u}{D^{\frac{1}{2}}} + \frac{dHNu^{\frac{1}{2}}}{Dq^{\frac{1}{2}}} + \left(\frac{Hqu}{dD} \right)^{\frac{1}{2}} \right),$$

hosil bo'ladi[32].

$L_2^{(2)}$ ni baholash uchun esa 3.3.4 lemmaning á) tasdig'ida

$$L_2^{(2)} \ll N^E \left(\frac{Hd^{\frac{1}{2}}N}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}}N^{\frac{1}{2}}q^{\frac{1}{4}}}{D^{\frac{3}{4}}d^{\frac{1}{2}}} \right).$$

Baho o'rinli bo'ladi.

$L_2^{(3)}$ ni esa trivial baholaymiz:

$$L_2^{(3)} \ll N^{1/2+\epsilon} HD^{-1/2}u$$

$L_2^{(4)}$ ni baholaymiz. Buning uchun 3.3.4-lemmaning á) qismida

$M_1 = N_2 = N^{\frac{1}{2}}, M_2 = u^2, N_1 = 1$ deb olamiz, u holda

$D^2 \leq N$ bo'lganda

$$L_2^{(4)} \ll N^k \left(\frac{HN^{\frac{1}{2}}u}{D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}}HN}{Dq^{\frac{1}{2}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}}q^{\frac{1}{4}}N^{\frac{1}{2}}}{d^{\frac{1}{4}}D^{\frac{3}{4}}} \right),$$

bajariladi.

Nihoyat 3.3.4-lemmaning b) qismida $M_1 = N_2 = N^{\frac{1}{2}}, M_2 = Nu^{-1}, N_1 = u$ deb olib $L_2^{(5)}$ yig'indi uchun quyidagi bahoga ega bo'lamiz:

$$L_2^{(5)} \ll N^\varepsilon \left(\frac{HN}{u^{\frac{1}{2}}D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}}HN}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}}q^{\frac{1}{4}}N^{\frac{1}{2}}}{d^{\frac{1}{4}}D^{\frac{3}{4}}} \right)$$

$L_2^{(f)}$, ($j=1,2,3,4,5$) lar uchun hosil bo'lgan baholarni yig'ib (3.3.17) dan

$$L_2 \ll N^k \left(\frac{HN}{u^{\frac{1}{2}}D^{\frac{3}{2}}} + \frac{HN^{\frac{1}{2}}u}{D^{\frac{1}{2}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{d^{\frac{1}{2}}HN}{d^{\frac{1}{4}}D} + \frac{H^{\frac{3}{4}}q^{\frac{1}{4}}N^{\frac{1}{2}}}{d^{\frac{1}{4}}D^{\frac{3}{4}}} + \frac{d^{\frac{1}{2}}HNu^{\frac{1}{2}}}{q^{\frac{1}{2}}D} + \left(\frac{Hqu}{Dd} \right)^{\frac{1}{2}} \right)$$

ni hosil qilamiz.

Endi L_3 ni baholash qoldi. Buning uchun esa $L_3(h)$ yig'indini bo'lakchalarga bo'lamiz: U holda L_3 ni quyidagicha yozish mumkin:

$$L_3 \leq \sum_{i=1}^3 \left(\sum_{h \leq H} L_3^{(j)}(h) \right) = \sum_{i=1}^3 L_3^{(i)}$$

Endi

$$\omega(m) = \tau_m = \left| \sum_{d \leq 4, d \mid m} \mu(d) \right| \leq \tau(m) \ll m^\varepsilon \text{ va } \psi(n) = \Lambda(n) = \ln n,$$

bo'lgani uchun 3.3.4-lemmaning á) qismida $M_1 = N_2 = N^{\frac{1}{2}}, M_2 = Nu^{-1}, N_1 = u$, deb olib $L_3^{(2)}$ yig'indi uchun

$$L_3^{(2)} \ll N^k \left(\frac{HN}{u^{\frac{1}{2}} D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}} HN}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{d^{\frac{1}{4}} D^{\frac{3}{4}}} \right)$$

bahoga ega bo‘lamiz. Bu baho $L_3^{(3)}$ yig‘inli uchun ham o‘rinli bo‘ladi.

$L_3^{(1)}$ ni baholash uchun esa $M_1 = N_1 = u$, $M_2 = N_2 = N^{\frac{1}{2}}$ deb olib 3.3.4-lemmaning á) qismini qo‘llaymiz. U holda

$$L_3^{(1)} \ll N^c \left(\frac{d^{\frac{1}{2}} HN}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{d^{\frac{1}{4}} D^{\frac{3}{4}}} \right),$$

ni hosil qilamiz. Demak,

$$L_3 \ll N^\varepsilon \left(\frac{HN}{u^{\frac{1}{2}} D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}} HN}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{d^{\frac{1}{4}} D^{\frac{3}{4}}} \right).$$

Shunday qilib, (3.3.17) dan

$$\sum_{k \in H} S_2(D, h\alpha, N) \ll N^\varepsilon \left(\frac{HN}{u^{\frac{1}{2}} D^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}} HN}{Dq^{\frac{1}{4}}} + \frac{HN^{\frac{7}{8}}}{D^{\frac{3}{4}}} + \frac{H^{\frac{3}{4}} q^{\frac{1}{4}} N^{\frac{1}{2}}}{d^{\frac{1}{4}} D^{\frac{3}{4}}} + \frac{HN^{\frac{1}{2}} u}{D^{\frac{1}{2}}} + \frac{HNu^{\frac{1}{2}}}{q^{\frac{1}{2}} D} + \left(\frac{Hqu}{Dd} \right)^{\frac{1}{2}} \right) \quad (3.3.19)$$

ekanligi kelib chiqadi.

Hozircha u bizda $u \leq N^{\frac{1}{3}}$ shartni qanoatlantiruvchi ixtiyoriy parameter edi. Endi, $u = \min \left[N^{\frac{1}{3}} D^{-\frac{5}{24}}; q^{\frac{1}{2}}; \left(N^2 H d q^{-1} D^{-1} \right)^{\frac{1}{2}} \right]$ deb olamiz. U holda

$q \leq N^2 H d D^{-1}$ ekanligini hisobga olib (3.3.17) dan 3.3.1-teoremaga ega bo‘lamiz.

3.3.1-teorema. Agar $|\alpha - aq^{-1}| \leq q^{-2}$, va $D^2 \leq N$ bo‘lsa, u holda ixtiyoriy soni uchun

$$\sum_{h < H} |S_2(D, h\alpha, N)| \ll HD^{-1}N^{1+\varepsilon} \left(d^2 q^{-1} + DN^{-\frac{1}{2}} + HqDN^{-2}d^{-1} \right)^{\frac{1}{4}}, \quad (3.3.1)$$

baho o'rinli. Bu yerda $d = (q, D)$ va Vinogradov simvoli «dagi doimiy faqat ε ga bog'liq.

Agar zarurat bo'lsa, (3.3.2) ning o'ng tomonidagi $\varepsilon > 0$ ni $\frac{c}{\ln \ln N}$ bilan almashtirish mumkin.

III bob uchun xulosalar

Uchinchi bobda Vinogradov teoremasi uning isboti va uni masalalarda o'rganildi. Tub sonlar bo'yicha olingan birinchi darajali trigonometrik yig'indilarning baholari olindi va uni additiv masalalar yechishga tatbiq etildi.

M. Vinogradov birinchi marta arifmetik progressiyaga tegishli tub sonlar bo'yicha olingan trigonometrik yig'indi

$$\sum_{\substack{p \leq N, p \equiv l \pmod{D} \\ (p, l) = 1}} e(ap)$$

moduli uchun trivial bo'lmagan baho olgan. Bu yig'indi quyidagi yig'indiga ekvivalent ekanligidan

$$S_1(\alpha) = S_1(D, \alpha, N) = \sum_{\substack{n \leq N \\ m \equiv l \pmod{D}}} \Lambda(n) e(\alpha n), \quad 1 \leq l \leq D, \quad (l, D) = 1$$

yig'indini modulini baholash uchun quyidagi ifodani baholadik

$$\sum_{x_q} |\psi(N, x_q)| \text{ va shunday natija oldik.}$$

$$\sum_{x_q} |\psi(N, x_q)| \ll (N + qN^{\frac{1}{2}} + q^\gamma N^\delta) L^c \text{ bundan esa}$$

$$S_1(\alpha) \ll \frac{N}{\varphi(D)} \Delta \text{ ushbu baho kelib chiqdi. Bu yerda } \Delta = DP^{-\frac{1}{2}} L^{\frac{11}{2}}.$$

XULOSA

Boblar bo'yicha umumiy amalga oshirilgan ishlardan quyidagilarga ega bo'ldik. Yuqoridagilardan kelib chiqqan holda quyidagi xulosalar qilishimiz mumkin. Trigonometrik yig'indilarning Matematikaning ko'pgina sohalariga xususan sonlar nazariyasida ahamiyati beqiyosdir. Trigonometrik yig'indilardan sonlar nazariyasining turli additiv masalalarini yechishda, xususan, qo'shiluvchilar soni chegaralangan bo'lgandagi Varing muammosini, Goldbax muammosini, Hardy –Littlvud muammosini, Xua–Lo–Ken muammosini hamda ularning umumlashmalarini yechishda foydalanish mumkin.

Ushbu ishda trigonometrik yig'indilarning berilgan oraliqdagi n-darajali chegirmalar sonini aniqlashga, eng kichik musbat boshlang'ich ildizni yuqoridan baholashga, natural sonlar bo'yicha olingan trigonometrik yig'indilarning ketma-ketliklar kasr qismlarining taqsimlanishini baholashga tadbirlari ko'rib chiqilgan

Ushbu

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}}$$

yig'indini qaraymiz.

Agarda S_1 yig'indida a soni m ga bo'linsa (a/m) , u holda $a = mt$ va

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i tx} = 1 + e^{2\pi i t} + e^{2\pi i (2t)} + \dots + e^{2\pi i (m-1)t} = \underbrace{1+1+\dots+1}_m = m,$$

bo'ladi. Agarda S_1 yig'indida a soni m ga bo'linmasa $(a \nmid m)$, u holda

$$S_1 = 1 + e^{2\pi i \frac{a}{m}} + \dots + e^{2\pi i \frac{a(m-1)}{m}} = \frac{1 - \left(e^{2\pi i \frac{a}{m}} \right)^m}{1 - e^{2\pi i \frac{a}{m}}} = \frac{1 - e^{2\pi i a}}{1 - e^{2\pi i \frac{a}{m}}} = 0.$$

Shunday qilib,

$$S_1 = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax}{m}} = \begin{cases} m, & \text{agar } a \mid m, \\ 0, & \text{agar } a \nmid m \end{cases}$$

ekanligini isbotladik.

Faraz etaylik avvalo q toq son bo'lsin. U holda

$$|S_2|^2 = e^{2\pi i \frac{a^2 0^2}{q}} \cdot q = q, \text{ ya'ni } |S_2| = \sqrt{q}.$$

Endi faraz etaylik q juft son bo'lsin. U holda $q = 2q_1$ deb olib

$$|S_2|^2 \leq 4q_1 = 2q \text{ yoki bundan } |S_2| \leq \sqrt{2q}$$

$$|S|^{2^{n-1}} \leq P^{2^{n-1}-1} + P^{2^{n-1}-n-\varepsilon_1} \sum_{y=1}^{n!p^{n-1}} \min\left(P, \frac{1}{(\alpha, y)}\right)$$

Veyl yig'indi uchun yuqoridagi yakuniy baho olindi

M. Vinogradov birinchi marta arifmetik progressiyaga tegishli tub sonlar bo'yicha olingan trigonometrik yig'indi

$$\sum_{\substack{p \leq N, p \equiv l \pmod{D} \\ (p, l) = 1}} e(ap)$$

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yig'indini modulini baholash uchun quyidagi ifodani baholadik

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$$\sum_{x_q} |\psi(N, x_q)| \ll (N + qN^{\frac{1}{2}} + q^\gamma N^\delta) L^c$$

bundan esa $S_1(\alpha) \ll \frac{N}{\varphi(D)} \Delta$ ushbu baho kelib chiqdi. Bu yerda

$$\Delta = DP^{-\frac{1}{2}} L^{\frac{11}{2}}.$$

Foydalanilgan adabiyotlar ro‘yhati:

1. O‘zbekiston Respublikasi Prezidentining “Ma‘muriy islohotlar doirasida oliy ta‘lim, fan va innovatsiyalar sohasida davlat boshqaruvini samarali tashkil qilish chora-tadbirlari to‘g‘risida” gi PQ-200-son qarori 03.07.2023.
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