# Kirish

**Magistrlik dissertatsiyasi mavzusining asoslanishi va uning dolzarbligi.**

Mamlakatimiz istiqlolga erishgan ilk kunlardanoq, davlatimiz tomonidan amalga oshirilayotgan bunyodkorlik ishlari vatanimiz mustaqilligi va ozodligi tufaylidir. Mustaqillik zamirida yuz berayotgan islohatlar sezilarli darajada insoniyat turmush tarzini o'zgartirib yubordi. So'ngi yillarda yoshlarga yaratilgan imkoniyatlar har bir yigit qizni harakatda bo'lishga undaydi.

Hozirgi kunda Prezidentimiz Sh.Mirziyoyev tomonidan "Mening nazarimda, jamiyat hayotining tanasi iqtisodiyot bo'lsa, uning joni va ruhi – ma'naviyatdir. Biz yangi O'zbekistonni barpo etishda ana shu ikkita mustahkam ustunga, ya'ni, bozor tamoyillariga asoslangan kuchli iqtisodiyotga hamda ajdodlarimizning boy merosi, milliy va umuminsoniy qadriyatlarga asoslangan kuchli ma'naviyatga tayanamiz"[1-4]. Yoshlarga keng imkoniyatlar yaratib berilmoqda, yirik loyihalar ustida ishlanmoqda. Ularning bilim va istedodlarini shakllantirib, milliy ma'naviyatimizni uzoqlashib ketayotgani sezilib qolmoqda. Ular o'zlari o'qib kelgan xorijiy davlatlardagi tajribani o'rganib, tajriba almashib kelishmoqda. Yangi O'zbekiston taraqqiyot strategiyasining maqsadi – aholining barcha qatlamlariga munosib hayot darajasini va turmush sharoitlarini yaratib berish, ishtimoiy himoya va bandlikni ta'minlash, daromadlar barqaror o'sishiga erishish, jamiyatning madaniy darajasi, bag'rikenglik va mehribonlik fazilatlarini yanada mustahkamlashdan iborat [5].

O'zbekistonda ta'lim tizimini isloh qilishning dasturiy hujjatlarida takidlanganidek, mamlakatimiz ta'lim tizimi hodimlari oldiga raqobatbardosh kadrlar tayyorlash, ta'lim tarbiya jarayonini jahon andozalari darajasiga yetkazishni asosiy vazifa qilib qo'ygan[6]. Shu ma'noda olib qaraganda, yoshlarning yangi avlodi istiqbol masalalarini kun tartibiga dadil qo'yadigan va uni yecha oladigan, siyosiy hamda ijtimoiy – iqtisodiy hayotda o'ziga mustaqil yo'l topa oladigan qobiliyatga ega bo'lishi kerak.

Hozirgi vaqtda O‘zbekistonda ilmiy va amaliy tatbiqlarga ega bo‘lgan fundamental fanlarga e’tibor yanada kuchaymoqda.

O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-son “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi to‘g‘risida”gi Farmoni, 2017-yil 17-fevraldagi PQ-2789-son “Fanlar akademiyasi faoliyati, ilmiy tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora tadbirlari to‘g‘risida”, 2017-yil 20-apreldagi PQ-2909-son “Oliy ta’lim tizimini yanada rivojlantirish chora-tadbirlari to‘g‘risida”, 2018-yil 27-apreldagi PQ-3682-son “Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliyotga joriy qilish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida”, 2020-yil 7-maydagi PQ-4708-son “Matematika sohasida ta’lim sifatini oshirish va ilmiy tadqiqot ishlarini rivojlantirish to‘g‘risida” gi qarorlari hamda fundamental fanlarga tegishli boshqa normativ huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda muayyan darajada xizmat qiladi.

**Tadqiqotning ob’ekti va predmeti:** Bir, ikki, to‘rt o‘lchovli gipergeometrik funksiyalar. Maxsus funksiyalar nazariyasi, matematik fizik tenglamalari, qatorlar nazariyasi, integro- differensial operatorlar.

**Tadqiqotning maqsadi va vazifalari:** Gamma va beta funksiyalarga, maxsus funksiyalarning, differensial operatorlarga oid asosiy tushunchalarni yoritish, 4 o‘lchovli gipergeometrik funksiyalarning yoyish formulalarini keltirish. Ikki o‘zgaruvchili uchunchi tartibli Campe de Feriet $F^{2;1;1}_{1;1;1}[x,y]$ gipergeometrik funktsiyasining integral ifodasini topish.

**Tadqiqotning ilmiy yangiligi** quyidagilardan iborat:

– 4 o‘lchovli gipergeometrik funksiyalarni yoyish formulalari topilgan;

– Campe de Feriet gipergometrik funksiyasining integral ifodasi topilgan.

**Tadqiqotning asosiy masalalari va farazlari.** Geometrik progressiyaning umumlashmasi bo'lgan gipergeometrik funksiya bir qator ajoyib xususiyatlarga ega bo'lib, bu funksiyalar ikki asr davomida matematiklarning e'tiborini tortdi. Ushbu funktsiyani o'rganish davomida Gaussni qatorlarning yaqinlashuvi masalasini, Rimanni esa  analitik davom ettirish muammosiga va maxsus nuqtalarga ega bo‘lgan differensial tenglamalarni o'rganishga olib keldi. Ushbu "gipergeometrik" atamasi fanga 1655 yilda Uollis tomonidan kiritilgan, keyinchalik esa Eyler va Kummer tomonidan o'rganilgan. Biroq, Gaussning ishidan oldin, bu qatorni so'zning zamonaviy ma'nosida funktsiya deb atash mumkin emas edi. Gauss gipergeometrik qatorning yaqinlashishini vaholanki gipergeometrik funktsiyaning mavjudligini isbotladi.

Biroq, Gauss ishidan keyin ham muammolar saqlanib qolindi. Gipergeometrik qator kompleks tekislikdagi birlik aylanadagina yaqinlashishini tushunish oson, gipergeometrik funktsiyani analitik ravishda bu doira chegarasidan tashqarida davom ettirish mumkin. Muammo gipergeometrik funktsiyaning butun kompleks tekisligiga analitik davomini qurishdir. Gipergeometrik funksiya uchun differensial tenglama yechimlarining xossalarini o‘rganish orqali bunday analitik davom ettirish mumkin.

**Tadqiqot mavzusi bo‘yicha adabiyotlar sharhi (tahlili)** Ma'lumki, zamonaviy matematika va nazariy fizikaning ko'plab muammolari ko'plab murakkab o'zgaruvchilarning gipergeometrik funktsiyalarini o'rganishga olib keladi. Bularga, masalan, supertor nazariyasi muammolari kiradi [Candelas P., de la Ossa X., Greene P. va Parkes L. Aynan eruvchan super konformal nazariya sifatida bir juft Calabi-Yau manifoldlari // Nucl. fizika. 1991. V. B539. P. 21-74.], Mellin-Barns tipidagi integrallarning analitik davomi [Passare M., Tsikh A.K., Cheshel A.A. Bir nechta Mellin-Barns integrallari bir nechta modulli Calabi-Yau manifoldlarining davrlari sifatida // Teor. va matematika. fizika. T. 109. 1996. B. 381-394.] va algebraik geometriya [Xorja R.P. Gipergeometrik funktsiyalar va torik navlardagi oyna simmetriyasi // Preprint. 1999. matematika. AG/9912109. B. 1-103.]. Gipergeometrik tipdagi differensial tenglamalar tizimlari zamonaviy kompyuter algebra tizimlarida qo‘llaniladigan ramziy hisob-kitoblar algoritmlarini amalga oshirish va disk raskadrovka qilishda notrivial model misollari sifatida keng qo‘llaniladi [Saito M., Sturmfels B. and

Takayama N. Grobner Deformations of Hypergeometric Differential Equations. Springer Verlag. Berlin, Heydelberg. 1999].

Dastlab, maxsus funktsiyalar bir-biriga bog'liq bo'lmagan ko'plab usullar bilan kiritilgan. Ularning paydo bo'lishi, qoida tariqasida, elementar funktsiyalar sinfida yechib bo'lmaydigan differentsial tenglamalarga (yoki bunday tenglamalar tizimlariga) olib keladigan muammolarni hal qilish zarurati bilan belgilandi. Bessel funksiyalari, Ermit funksiyalari va Gauss gipergeometrik funksiyasi shunday paydo bo'lgan. Keyinchalik, turli xil maxsus funktsiyalarni bog'laydigan juda ko'p sonli formulalarning paydo bo'lishi tabiiy ravishda mavjud turli xil maxsus funktsiyalarni tasniflash va tizimlashtirish istagini keltirib chiqardi. Gipergeometrik funktsiyalar matematik fizikaning maxsus funktsiyalari qatorida  muhim o'rin egallaydi.

**Tadqiqotda qo‘llanilgan metodikaning tavsifi.** Ko'p o'zgaruvchilarning gipergeometrik funksiyalari Knijnik-Zamolodchikov tenglamalarining yechimlari sifatida kvant maydon nazariyasida paydo bo'ladi [Virchenko A. Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups. Advanced Series in Mathematical Physics 21. World Scientific. 1995]. Bu tenglamalarni gipergeometrik tipdagi umumlashtirilgan tenglamalar deb hisoblash mumkin va ularning yechimlari bitta o‘zgaruvchining gipergeometrik funksiyalari uchun klassik Eyler integrallarini umumlashtiruvchi integral ifodalarini qabul qiladi. Bu yondashuv gipergeometrik tipdagi maxsus funksiyalarni va Li algebralari va kvant guruhlarini tasvirlash nazariyasidagi dolzarb masalalarni bog‘lash imkonini beradi.

**Tadqiqot natijalarining nazariy va amaliy ahamiyati.** Tadqiqot natijalarining ilmiy ahamiyati ko‘p o‘zgaruvchili gipergeometrik funksiyalar nazariyasining keyingi rivojida foydalanish mumkinligi, hamda tadqiqot natijalarining amaliy ahamiyati xususiy hosilali differensial tenglamalarga olib kelinadigan amaliy masalalarni yechishga tadqiq etilishi hamda maxsus nuqtalarga ega integral tenglamalarni yechishda qo‘llanishi bilan izohlanadi.

**Dissertatsiyaning tuzilishi va hajmi.** Dissertatsiya kirish, uchta bob, xulosa va adabiyotlar ro‘yxatidan va hajmi 55 betdan iborat.

## I B O B. GIPERGEOMETRIK FUNKSIYALARNI YOYISH FORMULALARINI O‘RGANISHNING NAZARIY ASOSLARI

### 1.1- §. Gamma va beta funksiyalar

***1. Gamma funksiyasi.***

Gamma funksiyasi Eylerning ikkinchi tur integrali orqali quyidagicha aniqlanadi [32], [23: 14 bet], [28]:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt=\int_0^1\left(\ln\frac{1}{t}\right)^{z-1}dt,\quad \operatorname{Re} z>0. \tag{1.1}$$

Gamma funksiyaning (1.1) ko`rinishi Eyler tomonidan kiritilgan.

$\Gamma(z)$ funksiyani boshqacha ko`rinishda ham yozib olish mumkin:

$$\Gamma(z)=z^{-1}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right], \tag{1.2}$$

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left[e^{-\frac{z}{n}}\left(1+\frac{z}{n}\right)\right],\quad \gamma=0,5772156649...\ , \tag{1.3}$$

bu yerda $\gamma-$Eyler-Maskeron o`zgarmasi. Gamma funksiyaning (1.2) va (1.3) ko`rinishi mos ravishda Gauss va Veyershtrass tomonidan kiritilgan.

(1.1) formulaga $t=su$ almashtirish bajarib, quyidagi

$$\Gamma(z)=s^{z}\int_0^\infty e^{-su}u^{z-1}du,\quad \operatorname{Re} z>0 \tag{1.4}$$

ko`rinishga keltiramiz, bu yerda $s-$ haqiqiy musbat o`zgarmas son.

Agar integrallash yo`li koordinata boshidan chiqib $Ox$ o`qi bilan $\delta$ burchak tashkil qilgan nurdan iborat bo`lsa, (1.4) formula $s$ kompleks son bo`lganda ham o`rinli bo`ladi:

$$\int_0^{\infty e^{i\delta}} e^{-su}u^{z-1}du=s^{-z}\Gamma(z),\quad \operatorname{Re} z>0,\ -\left(\frac{\pi}{2}+\delta\right)<arg\, s<\frac{\pi}{2}-\delta. \tag{1.5}$$

Agar $0<\operatorname{Re} z<1$ bo`lsa, u holda (1.5) formula $arg\, s+\delta=\pm\pi/2$ bo`lganda ham o`rinlidir.

Shuni aytish lozimki, (1.2) va (1.3) formulalarga ko`ra $z$ kompleks o`zgaruvchili $\Gamma(z)$ funksiya $z=0,-1,-2,\dots$ nuqtalardan tashqari butun kompleks tekislikda analitik funksiyadir [23: 15-16 betlar]. Bu funksiya $z=-n,$ $(n=0,1,2,...)$ nuqtalarda oddiy qutblarga hamda bu nuqtalarda mos ravishda $(-1)^n/n!$ [4: 1.17(11)]ga teng bo`lgan chegirmalarga ega bo`lgan meromorf funksiyadir. $\Gamma(z)$ funksiya kompleks tekislikda nollarga ega emas, demak $1/\Gamma(z)-$ butun funksiyadir.

***2. Gamma funksiyasining funksional munosabatlari.***

(1.1) formulada qatnashgan integralni bo`laklab integrallab, ushbu

$$u=e^{-t},\quad d\upsilon=t^{z-1}dt \ \Rightarrow\ du=-e^{-t}dt,\quad \upsilon=t^z/z$$

almashtirishlarni e'tiborga olib, quyidagiga

$$\Gamma(z)=\frac{e^{-t}}{z}t^z\Bigg|_{t=0}^{t=\infty}+\frac{1}{z}\int_0^\infty e^{-t}t^z dt=\frac{1}{z}\int_0^\infty e^{-t}t^z dt=\frac{1}{z}\Gamma(1+z)$$

ega bo`lamiz.

Demak, ushbu

$$\Gamma(1+z)=z\Gamma(z) \tag{1.6}$$

munosabat o`rinli.

Agar $n-$ natural son bo`lsa, u holda quyidagi

$$\Gamma(z+n)=z(z+1)(z+2)\dots(z+n-1)\Gamma(z) \tag{1.7}$$

funksional munosabat o`rinlidir.

(1.7) formuladan quyidagi

$$\frac{\Gamma(z)}{\Gamma(z-n)}=(z-1)(z-2)\dots(z-n)=(-1)^n\frac{\Gamma(-z+n+1)}{\Gamma(-z+1)}, \tag{1.8}$$

$$\frac{\Gamma(-z+n)}{\Gamma(-z)}=(-1)^n z(z-1)(z-2)\dots(z-n+1)=(-1)^n\frac{\Gamma(z+1)}{\Gamma(z-n+1)} \tag{1.9}$$

formulalar kelib chiqadi.

Agar (1.1) formulada $z=1$ bo`lsa, u holda

$$\Gamma(1)=\int_0^\infty e^{-t}dt=1 \qquad (1.10)$$

tenglik o`rinli bo`ladi.

(1.7) va (1.10) tengliklardan quyidagi

$$\Gamma(1+n)=1\cdot 2\cdot 3\ \ldots\cdot n=n! \qquad (1.11)$$

funksional munosabatni olamiz.

(1.3) ifodadan va

$$-\frac{1}{z}e^{\gamma z}\left\{\prod_{n=1}^{\infty}\left[e^{\frac{z}{n}}\left(1-\frac{z}{n}\right)\right]\right\}^{-1}=\Gamma(-z)$$

tenglikdan ushbu

$$\Gamma(z)\Gamma(-z)=-z^{-2}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)^{-1} \qquad (1.12)$$

ifodani olamiz. [30: 2-bo`lim 379 bet] kitobdagi ushbu

$$sin\,\pi z=z\pi\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

formuladan va (1.12) dan quyidagi

$$\Gamma(z)\Gamma(-z)=-\pi/z\,sin\pi z \qquad (1.13)$$

funksional bog'lanishni hosil qilamiz.

(1.13) va (1.6) formulalardan foydalanib, quyidagi

$$\Gamma(z)\Gamma(1-z)=\Gamma(z)(-z)\Gamma(-z)=\frac{z\pi}{zsin\pi z}=\frac{\pi}{sin\pi z}\quad\Rightarrow$$

$$\Gamma(z)\Gamma(1-z)=\pi/sin\pi z\,, \qquad (1.14)$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{cos\,\pi z} \qquad (1.15)$$

munosabatlarni olamiz.

Agar (1.14) va (1.1) formulalarda $z=1/2$ bo`lsa, u holda

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-u^2}du=\sqrt{\pi}, \qquad (1.16)$$

yoki

$$\Gamma^2\left(\frac{1}{2}\right)=\frac{\pi}{sin(\pi/2)}=\pi \quad \Rightarrow \quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

tenglik o`rinli bo`ladi.

Gauss – Lejandrning ko`paytma formulasi:

$$\prod_{r=0}^{m-1}\Gamma\left(z+\frac{r}{m}\right)=(2\pi)^{\frac{1}{2}(m-1)}\, m^{\frac{1}{2}-mz}\Gamma(mz). \qquad (1.17)$$

Agar (1.17) formulada $m=2$ bo`lsa, u holda quyidagi

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \qquad (1.18)$$

Lejandr formulasi o`rinlidir.

***3. Beta - funksiyasi.*** $B(p,q)$ beta-funksiyasi quyidagi

$$\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad \operatorname{Re} p>0, \quad \operatorname{Re} q>0 \qquad (1.19)$$

ko`rinishdagi Eylerning birinchi tur integrali yordamida aniqlanadi [23: 23-28 betlar].

(1.19) formulaga $t=\upsilon/(1+\upsilon)$ almashtirish bajarib, ushbu

$$\mathrm{B}(p,q)=\int_0^\infty \upsilon^{p-1}(1+\upsilon)^{-p-q}d\upsilon, \quad \operatorname{Re} p>0, \quad \operatorname{Re} q>0 \qquad (1.20)$$

$$\mathrm{B}(p,q)=\int_0^1 (\upsilon^{p-1}+\upsilon^{q-1})(1+\upsilon)^{-p-q}d\upsilon, \quad \operatorname{Re} p>0, \quad \operatorname{Re} q>0 \qquad (1.21)$$

formulalarga ega bo`lamiz.

(1.21) tenglikdan quyidagi

$$\mathrm{B}(p,q)=\mathrm{B}(q,p) \qquad (1.22)$$

ayniyat kelib chiqadi.

$B(p,q)$ beta-funksiyasi gamma funksiya orqali quyidagicha

$$B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q) \qquad (1.23)$$

ifodalanadi. Haqiqatan, (1.4) formulaga asosan ushbu

$$\Gamma(p+q)=s^{p+q}\int_0^\infty e^{-su}u^{p+q-1}du, \quad \mathrm{Re}(p+q)>0$$

tenglikni olamiz. Bunda $s=1+\upsilon$ deb qabul qilsak, quyidagi

$$\int_0^\infty e^{-(1+\upsilon)u}u^{p+q-1}du=\frac{\Gamma(p+q)}{(1+\upsilon)^{p+q}} \tag{1.24}$$

ayniyatni ega bo`lamiz.

Endi (1.24) ayniyatni ikkala tomonini $\upsilon^{p-1}$ ga ko`paytirib, $\upsilon$ bo`yicha 0 dan $\infty$ gacha integrallab, so`ng integrallash tartibini almashtiramiz:

$$\int_0^\infty \upsilon^{p-1}d\upsilon\int_0^\infty e^{-(1+\upsilon)u}u^{p+q-1}du=\Gamma(p+q)\int_0^\infty \upsilon^{p-1}(1+\upsilon)^{p+q}d\upsilon \quad \Rightarrow$$

$$\int_0^\infty e^{-u}u^{q-1}du\int_0^\infty e^{-\upsilon u}(u\,\upsilon)^{p-1}d(u\upsilon)=\Gamma(p+q)\int_0^\infty \upsilon^{p-1}(1+\upsilon)^{p+q}d\upsilon.$$

Bundan, (1.1) va (1.20) formulalarga ko`ra quyidagi

$$\Gamma(p)\Gamma(q)=\Gamma(p+q)B(p,q) \quad \text{yoki} \quad B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$$

ayniyatni olamiz. Demak, (1.23) ayniyat o`rinlidir.

***4.***

***Beta- funksiyasining funksional munosabatlari.*** $B(p,q)$ beta-funksiya uchun quyidagi funksional munosabatlar o`rinlidir:

$$B(p,q+1)=\frac{q}{p}B(p+1,q)=\frac{q}{p+q}B(p,q), \tag{1.25}$$

$$B(p,q)B(p+q,l)=B(q,l)B(q+l,p)=B(l,p)B(p+l,q), \tag{1.26}$$

$$B(p,q)B(p+q,l)B(p+q+l,z)=\frac{\Gamma(p)\Gamma(q)\Gamma(l)\Gamma(z)}{\Gamma(p+q+l+z)}, \tag{1.27}$$

$$\frac{1}{B(n,m)}=mC_{n+m-1}^{n-1}=nC_{n+m-1}^{n-1}, \quad m,n=1,2,\dots, \quad C_\beta^\alpha=\frac{\Gamma(1+\beta)}{\alpha!\Gamma(1+\beta-\alpha)}. \tag{1.28}$$

***5. Beta-funksiyasining aniq integrallar yordamida ifodalanishi.*** Aniq integrallar yordamida $B(p,q)$ beta funksiyasi quyidagicha ifodalanadi:

$$\int_0^1 t^{p-1}(1-t)^{q-1}(1+bt)^{-p-q}dt=(1+b)^{-p}\mathrm{B}(p,q), \quad b>-1,\ \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0, \tag{1.29}$$

$$\int_0^\infty t^{p-1}(1+bt)^{-p-q}dt = b^{-p}\mathrm{B}(p,q), \quad b>0,\ \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0, \tag{1.30}$$

$$\int_b^a (t-b)^{p-1}(a-t)^{q-1}dt = (a-b)^{p+q-1}\mathrm{B}(p,q), \quad b<a,\ \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0, \tag{1.31}$$

$$\int_0^1 t^{p-1}\left(1-t^z\right)^{q-1}dt = z^{-1}\mathrm{B}\left(pz^{-1},q\right), \quad z>0,\ \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0, \tag{1.32}$$

$$\int_0^{\pi/2} (sin t)^{2p-1}(cos t)^{2q-1}dt = 2^{-1}\mathrm{B}(p,q), \quad \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0, \tag{1.33}$$

$$\int_0^\infty (sht)^p(cht)^{-q}dt = 2^{-1}\mathrm{B}\left(\frac{p+1}{2},\frac{q-p}{2}\right), \quad \mathrm{Re}\,p>-1,\ \mathrm{Re}(p-q)<0, \tag{1.34}$$

$$\int_0^\infty e^{-pt}\left(1-e^{-tz}\right)^{q-1}dt = z^{-1}\mathrm{B}\left(\frac{p}{z},q\right), \quad \mathrm{Re}\frac{p}{z}>0,\ \mathrm{Re}\,p>0,\ \mathrm{Re}\,q>0. \tag{1.35}$$

## 1. 2- §. Gaussning gipergeometrik funksiyasi haqida tushuncha

### 1. 2. 1. Gauss tenglamasini yechish. Kummer yechimlari

**1. Gauss tenglamasini yechish.** Ushbu

$$x(1-x)y''+\big(c-(a+b+1)x\big)y'-aby=0 \tag{1.36}$$

tenglamaga gipergeometrik tenglama yoki **Gauss tenglamasi** deyiladi, bu erda $a, b, c$ -berilgan o`zgarmas sonlar bo`lib, ular ixtiyoriy kompleks yoki haqiqiy sonlar bo`lishi mumkin. (1.36) tenglama uchta maxsus nuqtalarga ega, ya'ni umumiylikka ziyon yetkazmagan holda ularni $0,\ \infty,\ 1$ nuqtalardan iborat deb olish
mumkin[29: 10.3 - band].

(1.36) tenglamaning $x=0$ maxsus nuqta atrofidagi yechimini

$$y = x^{\rho}\sum_{i=0}^{\infty} a_i x^i,\ (a_0 \neq 0) \qquad (1.37)$$

ko`rinishda izlaymiz.

Gauss tenglamasi uchun aniqlovchi tenglama ($a_0 \neq 0$ bo`lgani uchun) $\rho(\rho-1)+c\rho=0$ ko`rinishga ega bo`lib, bundan $\rho_1 = 0$ va $\rho_2 = 1-c$. Demak, (1.36) tenglamada, $\rho$ ning $\rho_1 = 0$ qiymatiga mos birinchi xususiy yechimi ushbu

$$y_1 = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ... \qquad (1.38)$$

musbat darajali qator ko`rinishida bo`ladi.

Izlanayotgan (1.38) yechimning kerakli tartibli hosilalarini hisoblab, (1.36) ga qo`yamiz va $x^k$ ning oldidagi koeffitsientini nolga tenglashtiramiz:

$$(k+1)(c+k)a_{k+1} - (a+k)(b+k)a_k = 0,$$

bundan

$$a_{k+1} = \frac{(a+k)(b+k)}{(k+1)(c+k)} a_k, \quad k = 0,1,2,3,... . \qquad (1.39)$$

$a_0$ ixtiyoriy va $a_0 \neq 0$ bo`lgani uchun, umumiylikka ziyon yetkazmay $a_0 = 1$ deb olamiz, hamda (1.39) dan noma'lum koeffitsientlarni quyidagi

$$a_1 = \frac{ab}{c},\ a_2 = \frac{(a+1)(b+1)ab}{1\cdot 2\cdot c(c+1)},\ a_3 = \frac{(a+2)(b+2)(a+1)(b+1)ab}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)},...,$$

$$a_k = \frac{(a+k-1)(b+k-1)(a+k-2)(b+k-2)\cdot ...\cdot (a+1)(b+1)ab}{1\cdot 2\cdot ...\cdot k(c+k-1)(c+k-2)\cdot ...\cdot (c+1)c}.$$

ko`rinishda topamiz.

Shuni ta'kidlash lozimki, noma'lum koeffitsientlar aniq topilishi uchun $c$ nol va manfiy butun son bo`lmasligi kerak, ya'ni $c \neq 0, -1, -2, ...$ .

Demak, topilgan koeffitsientlarni (1.38) ga qo`yib, (1.36) tenglamaning birinchi xususiy yechimini quyidagi ko`rinishda topamiz:

$$y_1 = F(a,b,c;x) = 1 + \frac{ab}{c}x + \frac{ab(a+1)(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \ldots +$$

$$+\frac{(a+k)(b+k)(a+k-1)(b+k-1)\cdot\ldots\cdot(a+1)(b+1)ab}{1\cdot 2\cdot\ldots\cdot k(k+1)(c+k)(c+k-1)\cdot\ldots\cdot(c+1)c}\; x^{k+1} + \ldots \quad (1.40)$$

Bu (1.40) yechimga **Gaussning gipergeometrik qatori** deyiladi.

(1.40) qatorda ushbu

$$(\sigma)_0 = 1, \quad (\sigma)_n = \sigma(\sigma+1)(\sigma+2)\cdots(\sigma+n-1) = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} \qquad (1.41)$$

belgilashlarni kiritib, uni

$$F(a,b,c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\, x^n \qquad (1.42)$$

ko`rinishda yozib olamiz, bu yerda $(a)_n$ – Poxgammer belgisi deyiladi, u (1.41) formula orqali aniqlanadi.

(1.42) qator $|x|<1$ doirada absolyut va tekis yaqinlashadi.

Raabe alomatiga ko`ra [30: 2-bo`lim 275 bet] (1.42) **Gaussning gipergeometrik qatori** uchun quyidagi tasdiqlar o`rinlidir:

1) agar $\mathrm{Re}(c-a-b)>0$ bo`lsa, u holda (1.42) qator $|x|=1,\ x\neq 1$ aylanada **absolyut va tekis** yaqinlashadi;

2) agar $-1<\mathrm{Re}(c-a-b)\le 0$ bo`lsa, u holda (1.42) qator $|x|=1$ aylanada **shartli** yaqinlashadi;

3) agar $\mathrm{Re}(c-a-b)\le -1$ bo`lsa, u holda (1.42) qator $|x|=1$ aylanada **uzoqlashuvchi** bo`ladi.

(1.36) tenglamaning $\rho_2 = 1-c$ ga nisbatan ikkinchi xususiy yechimini topishdan avval, (1.36) tenglamada

$$y(x) = x^{1-c}u(x) \qquad (1.42_1)$$

almashtirish bajarib, bu tenglamani

$$x(1-x)u'' + \left(2-c-(a+b-2c+3)x\right)u' - (a+1-c)(b+1-c)u = 0 \quad (1.43)$$

ko`rinishda yozib olamiz. U holda (1.36) tenglamadagi $a, b$ va $c$ parametrlar mos ravishda $a+1-c,\ b+1-c$ va $2-c$ parametrlarga o`zgaradi. Demak, (1.43) tenglamaning bir xususiy yechimi

$$u_1 = F(a+1-c, b+1-c, 2-c; x)$$

ko`rinishda bo`ladi.

Shunday qilib, $(1.42_1)$ ga asosan (1.36) tenglamaning ikkinchi xususiy yechimi quyidagicha

$$y_2 = x^{1-c} F(a+1-c,\ b+1-c,\ 2-c;\ x) \qquad (1.44)$$

topiladi, bu yerda $2-c \neq 0, -1, -2, \ldots$ .

Xullas, $c-$butun son bo`lmaganda, (1.36) tenglamaning umumiy yechimi

$$y = C_1 F(a, b, c; x) + C_2 x^{1-c} F(a+1-c,\ b+1-c,\ 2-c;\ x) \qquad (1.45)$$

ko`rinishda bo`ladi, bu yerda $C_1$ va $C_2$ ixtiyoriy o`zgarmas sonlardir.

**Eslatma.** Agar (1.36) tenglamada $c-$butun son bo`lsa, aniqlovchi tenglama ildizlari orasidagi ayirma nol yoki butun son bo`ladi, bu holda (1.36) tenglama umumiy yechimida logarifmik had qatnashadi.

(1.36) tenglamani $x=1$ maxsus nuqta atrofidagi yechimini hosil qilish uchun $x$ ni $1-x$ ga almashtirish yetarlidir. Unda (1.36) tenglamaning parametrlari mos ravishda $a,\ b$ va $1+a+b-c$ parametrlarga o`zgardi. Bu holda (1.36) tenglamani $x=1$ maxsus nuqta atrofidagi xususiy yechimlari ushbu

$$y_3(x) = F(a,\ b,\ 1+a+b-c; 1-x), \qquad (1.46)$$

$$y_4(x) = (1-x)^{c-a-b} F(c-a,\ c-b,\ 1+c-a-b; 1-x) \qquad (1.47)$$

ko`rinishda bo`ladi, bu yerda $c-a-b$ butun sonlar bo`lmasligi kerak va $|arg(1-x)| < \pi$ .

(1.36) tenglamani $x=\infty$ maxsus nuqta atrofidagi yechimini hosil qilish uchun $x$ ni $1/x$ ga almashtirish yetarlidir. Unda (1.36) tenglamaning parametrlari

mos ravishda $a,\ b+a-c$ va $1+a-b$ parametrlarga o`zgardi. Bu holda (1.36) tenglamani $x=\infty$ maxsus
nuqta atrofidagi xususiy yechimlari quyidagi

$$y_5(x)=z^{-a}F\left(a,\ 1+a-c,\ 1+a-b;1/x\right), \qquad (1.48)$$

$$y_6(x)=x^{-b}F\left(b,\ 1+b-c,\ 1-a+b;1/x\right) \qquad (1.49)$$

ko`rinishda aniqlanadi, bu yerda $a-b$ butun sonlar bo`lmasligi kerak va $|arg(-x)|<\pi$ .

**2. Kummer yechimlari.**

Shunday qilib, (1.36) Gauss tenglamasining asosiy 6 ta xususiy yechimini gipergeometrik funksiyalar yordamida yozib oldik. Bu yechimlardan tashqari **Kummer yechimlari** ham mavjud bulardan ayrimlarini keltiramiz:

$$y_1(x)=F(a,b,c,x)=(1-x)^{c-a-b}F(c-a,\,c-b,\,c;\,x)= \qquad (1.50)$$

$$=(1-x)^{-a}F\left(a,c-b,c;\,x/(x-1)\right)=(1-x)^{-b}F\left(c-a,b,c;\,x/(x-1)\right), \quad (1.51)$$

$$y_3(x)=F(a,\ b,\ 1+a+b-c;1-x)=$$

$$=z^{1-c}F(a+1-c,\ b+1-c,\ 1+a+b-c;1-x)= \qquad (1.52)$$

$$=z^{-a}F(a,\ a+1-c,\ 1+a+b-c;1-x^{-1})= \qquad (1.53)$$

$$=z^{-b}F(b+1-c,b,\ 1+a+b-c;1-x^{-1}), \qquad (1.54)$$

$$y_5(x)=(-z)^{-a}F\left(a,\ 1+a-c,\ 1+a-b;\,x^{-1}\right)=$$

$$=(-z)^{b-c}(1-x)^{c-a-b}F\left(1-b,\ c-b,\ 1+a-b;\ x^{-1}\right)= \qquad (1.55)$$

$$=(1-x)^{-a}F\left(a,\ c-b,\ 1+a-b;\ (1-x)^{-1}\right)= \qquad (1.56)$$

$$=(-z)^{1-c}(1-x)^{c-a-1}F\left(1+a-c,\ 1-b,\ 1+a-b;\ (1-x)^{-1}\right). \qquad (1.57)$$

**3.** $F(a,b,c,x)$ **gipergeometrik funksiyaning sodda xossalari.** Gipergeometrik funksiyaning sodda xossalari (1.42) qatordan kelib chiqadi.

**a)** $F(a,b,c,x)$ gipergiometrik funksiya $a$ va $b$ parametrlarga nisbatan simmetrikdir, ya'ni

$$F(a, b, c, x) = F(b, a, c, x)\,; \tag{1.58}$$

**b)** agar $b = c$ bo`lsa, u holda quyidagi

$$F(a, b, b, x) = (1-x)^{-a}, \qquad |arg(1-x)| < \pi \tag{1.59}$$

tenglikka ega bo`lamiz;

**c)** agar $a = -n$ yoki $b = -n,\quad n = 0,1,2,...$ bo`lsa, u holda (1.42) darajali qator uziladi va u quyidagi

$$F(-n,b,c,x) = \sum_{r=0}^{n} \frac{(-n)_r (b)_r}{r!(c)_r} x^r \quad \text{yoki} \quad F(a,-n,c,x) = \sum_{r=0}^{n} \frac{(a)_r (-n)_r}{r!(c)_r} x^r \tag{1.60}$$

ko`rinishni oladi;

**d )** $F(a,b,c,x)$ gipergeometrik funksiya uchun quyidagi

$$|F(a,b,c,x)| \le \begin{cases} const, & \mathrm{Re}(c-a-b) > 0, \quad 0 \le x \le 1, \\ const(1-x)^{c-a-b}, & \mathrm{Re}(c-a-b) < 0, \quad 0 < x < 1, \\ const\left[1+\ln(1-x)\right], & \mathrm{Re}(c-a-b) = 0, \quad 0 < x < 1 \end{cases} \tag{1.61}$$

baho o`rinlidir[43];

**e )** $F(a,b,c,x)$ gipergeometrik funksiya uchun ushbu

$$F(a,b,c,0) = F(0,b,c,x) = F(a,0,c,x) = 1 \tag{1.62}$$

tenglik o`rinli.

### 1.3-§. Gipergeometrik funksiyaning integral ifodalari

Agar $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$, $|arg(1-x)| < \pi$ bo`lsa, u holda gipergeometrik funksiya uchun ushbu

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt \tag{1.63}$$

**integral ifoda** (Eyler formulasi) o`rinlidir [32].

(1.63) tenglikni isbotlash uchun $(1-xt)^{-a}$ funksiyani $xt$ ning darajalari bo`yicha binomial qatorga yoyamiz va bu yoyilmani $t^{b-1}(1-t)^{c-b-1}$ ifodaga ko`paytirib $t$ bo`yicha 0 dan 1 gacha hadma-had integrallaymiz, ya'ni

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt = \sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt. \qquad (1.64)$$

(1.19), (1.23) va (1.41) formulalarga ko`ra (1.64) tenglikni quyidagicha hisoblaymiz:

$$\sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n B(b+n,c-b) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n \times$$

$$\times\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \sum_{n=0}^{\infty}\frac{(a)_n x^n}{n!}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} =$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c;x).$$

Shunday qilib, oxirgi tenglik va (1.64) formuladan (1.63) tenglikni to`g'riligi kelib chiqadi.

Gipergeometrik funksiya uchun (1.63) integral ifodadan tashqari quyidagi integral ifodalar ham o`rinlidir:

$$F(a,b,c;1-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^{\infty} t^{b-1}(1+t)^{a-c}(1+xt)^{-a}dt,$$

$$\mathrm{Re}\,c > \mathrm{Re}\,b > 0, \ |arg\, x| < \pi, \qquad (1.65)$$

$$F(a,b,c;x^{-1}) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^{\infty} t^{a-c}(t-1)^{c-b-1}(t-x^{-1})^{-a}dt,$$

$$1+\mathrm{Re}\,a > \mathrm{Re}\,c > \mathrm{Re}\,b > 0, \ |arg\,(x-1)| < \pi, \qquad (1.66)$$

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}F(a,b,c;xt)dt,$$

$$\mathrm{Re}\,c > \mathrm{Re}\,b > 0, \ x \neq 1, \ |arg(1-x)| < \pi. \qquad (1.67)$$

Gipergeometrik funksiya uchun integral ifodalar haqidagi to`liq ma'lumotni [23: 2.4 va 2.12 bandlar, 89 va 123 betlar] kitobdan olish mumkin.

(1.63) formulada $\operatorname{Re} c > \operatorname{Re} b > 0$, $\operatorname{Re}(c-a-b) > 0$, $c \neq 0,-1,-2,...$ bo`lib $x=1$ bo`lsa, u holda ushbu tenglik o`rinli:

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \qquad (1.68)$$

Haqiqatan ham, (1.63) formulada $x=1$ bo`lsa, u holda (1.29) va (1.23) ko`ra uni quyidagi ko`rinishda topamiz:

$$F(a,b,c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1} dt =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Bu esa (1.68) tenglikni to`g'riligini isbotlaydi.

### 1. 2. 3. Gipergeometrik funksiyani analitik davom ettirish

(1.36) tenglamaning $|x|<1$doirada aniqlangan regulyar yechimi $F(a,b,c,x)$ funksiyani $x$ o`zgaruvchining butun kompleks tekisligiga analitik davom ettirish mumkin. $F(a,b,c;x)$ funksiyani analitik davom ettirish (1.63) integral ifoda yordamida amalga oshirish mumkin. Haqiqatan ham, (1.63) integral ifodada ushbu $t=(1-s)/(1-xs)$ almashtirishni bajarib, quyidagi

$$1-t = \frac{s(1-x)}{1-xs}, \quad 1-tx = \frac{1-x}{1-xs}, \quad dt = -\frac{(1-x)ds}{(1-xs)^2}$$

tengliklarni e'tiborga olib,

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b,c,x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt = (1-x)^{c-a-b} \times$$

$$\times \int_0^1 \frac{s^{c-b-1}(1-s)^{b-1}}{(1-xs)^{c-a}} ds = (1-x)^{c-a-b} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(c-a,c-b,c;x)$$

tenglikni hosil qilamiz.

Bundan esa, ushbu

$$F(a,b,c;x)=(1-x)^{c-a-b}F(c-a,c-b,c;x),\ \operatorname{Re}c>\operatorname{Re}b>0,\ \ |arg(1-x)|<\pi. \quad (1.69)$$

formulani hosil qilamiz.

(1.69) formulaga **avtotransformatsiya formulasi** deyiladi.

Endi (1.36) tenglamaning (1.46) va (1.47) chiziqli erkli yechimlaridan foydalanib, $F(a,b,c,x)$ funksiyani bu yechimlarning chiziqli kombinatsiyasi orqali analitik davom ettirish mumkin:

$$F(a,b,c,x)=C_3F(a,\ b,\ 1+a+b-c;1-x)+C_4(1-x)^{c-a-b}\times$$
$$\times F(c-a,\ c-b,\ 1+c-a-b;1-x),$$
$$c-a-b\neq\ 0,\ \pm1,\ \pm2,...,\qquad |arg(1-x)|<\pi, \qquad (1.70)$$

bu yerda $C_3$, $C_4-$ koeffitsientlar $a,\ b,\ c$ parametrlarning analitik funksiyasidir. Bu koeffitsientlarni topish $\operatorname{Re}(c-a-b)$ ifodaning ishorasiga bog'liq, chunki (1.70) tenglikning o`ng tomoni $x=1$ nuqtada $a,\ b,\ c\ (c\neq n)$ parametrlarning ixtiyoriy qiymatida ma'noga ega, uning chap tomonining chekliligi $\operatorname{Re}(c-a-b)$ ifodaning ishorasiga bog'liq((1.68)formulaga qarang).

1) Agar $\operatorname{Re}(c-a-b)>0$ bo`lsa, u holda (1.68) formula o`rinli, ya'ni

$$C_3=F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (1.68)$$

tenglik o`rinli.

2) Agar $\operatorname{Re}(c-a-b)<0$ bo`lsa, u holda (1.70) tenglikning chap tomoniga (1.69) avtotransformatsiya formulasini qo`llab, so`ng $(1-x)^{a+b-c}$ ifodaga ko`paytirib, $x=1$ nuqtada $C_4$ koeffitsient

$$C_4=F(c-a,c-b,c;1)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \qquad (1.71)$$

ko`rinishda topiladi.

(1.68) va (1.71) koeffitsientlarni (1.70) tenglikga qo`yib, ushbu

$$F(a,b,c,x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,\ b,\ 1+a+b-c;1-x)+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}F(c-a,\ c-b,\ c-a-b+1;1-x),$$

$$c-a-b\neq 0,\ \pm 1,\ \pm 2,\dots,\qquad |arg(1-x)|<\pi, \tag{1.72}$$

Bolts formulasini hosil qilamiz.

(1.72) formula $F(a,b,c;x)$ gipergeometrik funksiyani $|x|<1$ sohadan $|1-x|<1,\ |arg(1-x)|<\pi$ sohaga **analitik davomini** beradi.

Endi (1.63) integral ifodaga ushbu

$$t=1-s,\quad 1-t=s,\quad 1-xt=1-(1-s)x=(1-x)\left(1-\frac{x}{x-1}\right),\quad dt=-ds$$

almashtirishni bajarib, quyidagi

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c,x)=\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}dt=(1-x)^{-a}\int_0^1\frac{s^{c-b-1}}{(1-s)^{1-b}}\times$$

$$\times\left(1-\frac{xs}{x-1}\right)^{-a}ds=(1-x)^{-a}\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F\left(c-a,b,c;\frac{x}{x-1}\right)$$

tenglikka ega bo`lamiz. Bundan esa, ushbu

$$F(a,b,c;x)=(1-x)^{-a}F\left(c-a,b,c;x/(x-1)\right). \tag{1.73}$$

formulani hosil qilamiz.

Shuni aytish lozimki, agar $\operatorname{Re}x<1/2$ bo`lsa, u holda $|x/(x-1)|<1$ tengsizlik o`rinli bo`ladi. Shuning uchun, (1.73) formula $F(a,b,c;x)$ gipergeometrik funksiyani $|x|<1$ sohadan $\operatorname{Re}x<1/2$ yarim tekislikka **analitik davomini** beradi.

(1.73) formulada $x$ ni $1-x$ ga almashtirsak, (1.73) formulani ushbu

$$F(a,b,c;1-x)=x^{-a}F\left(c-a,\ b,\ c;1-x^{-1}\right). \tag{1.74}$$

ko`rinishda yozib olamiz.

(1.72) va (1.74) formulalarni ketma-ket qo`llab, quyidagi

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}x^{-a}F\left(a,\ a-c+1,\ 1+a+b-c;1-x^{-1}\right)+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}x^{a-c}(1-x)^{c-a-b}F\left(c-a,\,1-a,\,c-a-b+1;1-x^{-1}\right),$$

$$c-a-b\neq 0,\ \pm 1,\ \pm 2,\ldots,\ \ |argx|<\pi,\quad |arg(1-x)|<\pi \qquad (1.75)$$

tenglikni olamiz.

(1.73) formula $F(a,b,c;x)$ gipergeometrik funksiyani $|x|<1$ doiradan $\operatorname{Re}x>0{,}5$ sohaga **analitik davom** etirish imkonini beradi.

Xuddi (1.70) formulaga o`xshash (1.36) tenglamaning (1.48) va (1.49) chiziqli erkli yechimlaridan foydalanib, $F(a,b,c;x)$ funksiyani bu yechimlarning chiziqli kombinatsiyasi orqali analitik davomini quyidagi

$$F(a,b,c,x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}F(a,\,1-c+a,\,1+a-b;\,x^{-1})+$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}F(b,\,1-c+b,\,1-a+b;\,x^{-1}),$$

$$a-b\neq 0,\ \pm 1,\ \pm 2,\ldots,\ \ |arg(1-x)|<\pi,\quad |arg(-x)|<\pi \qquad (1.76)$$

ko`rinishda topish mumkin.

(1.76) formula $F(a,b,c;x)$ gipergeometrik funksiyani $|x|<1$ sohadan $|x|>1$ sohaga **analitik davomini** beradi.

(1.76) va (1.73) formulalarni ketma-ket qo`llab, quyidagi

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-x)^{-a}F\left(a,\,c-b,\,1+a-b;(1-x)^{-1}\right)+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\times$$

$$\times(1-x)^{-b}F\left(b,c-a,1-a+b;(1-x)^{-1}\right),\ \ a-b\neq 0,\ \pm 1,\ \pm 2,\ldots,\ |arg(1-x)|<\pi \qquad (1.77)$$

tenglikni hosil qilamiz.

(1.77) formula $F(a,b,c;x)$ gipergeometrik funksiyani $|x|<1$ sohadan $|1-x|>1,\ \ |arg(1-x)|<\pi$ sohaga **analitik davomidir**.

$F(a,b,c;x)$ gipergemetrik funksiyani **logarifmik**, ya'ni $\operatorname{Re}(c-a-b)=0$ bo`lgan holdagi **analitik davomi** ushbu

$$F(a,b,a+b;x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}F(a,b,1;1-x)\ln(1-x)+\frac{\Gamma(a+b)}{\Gamma^2(a)\Gamma^2(b)}\times$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{(k!)^2}\left[\frac{2\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(a+k)}{\Gamma(a+k)}-\frac{\Gamma'(b+k)}{\Gamma(b+k)}\right](1-x)^k,$$

$$a,b\neq 0,\ -1,\ -2,\dots,\qquad |arg(1-x)|<\pi \qquad (1.78)$$

ko`rinishda bo`ladi, bu yerda $\frac{\Gamma'(x)}{\Gamma(x)}=\frac{d\ln\Gamma(x)}{dx}$ – gamma funksiyasining logarifmik hosilasi.

### 1.2.4. Gipergeometrik funksiyalari orasidagi asosiy bog’lanish formulalari

Yuqorida keltirilgan gipergeometrik funksiya xossalariga ko`ra ushbu $F(a\pm1)\equiv F(a\pm1,b,c;x),\quad F(b\pm1)\equiv F(a,b\pm1,c;x),\quad F(c\pm1)\equiv F(a,b,c\pm1;x)$ oltita funksiyalar $F\equiv F(a,b,c;x)$ funksiya bilan **yondosh funksiyalar** deyiladi.

Bu funksiyalarning xarakteriga asosan shuni aytish mumkinki, $F(a,b,c;x)$ va unga ixtiyoriy ikkita yondosh funksiyalar orasida koeffitsientlari $x$ga nisbatan chiziqli bo`lgan bog’liqlik mavjud. Shunday chiziqli bog’liqliklarning 15 tasini **birinchi bo`lib Gauss** tomonidan topilgan[23: 2.8 (31) - 2.8 (45)]:

$$(b-a)F+aF(a+1)-bF(b+1)=0, \qquad (1.79)$$

$$(c-a-1)F+aF(a+1)-(c-1)F(c-1)=0, \qquad (1.80)$$

$$(c-b-1)F+bF(b+1)-(c-1)F(c-1)=0, \qquad (1.81)$$

$$(c-a-b)F+a(1-x)F(a+1)-(c-b)F(b-1)=0, \qquad (1.82)$$

$$(c-a-b)F+b(1-x)F(b+1)-(c-a)F(a-1)=0, \qquad (1.83)$$

$$(b-a)(1-x)F+(c-b)F(b-1)-(c-a)F(a-1)=0, \qquad (1.84)$$

$$c(1-x)F+(c-b)\,x\,F(c+1)-cF(a-1)=0, \qquad (1.85)$$

$$c(1-x)F+(c-a)\,x\,F(c+1)-cF(b-1)=0, \qquad (1.86)$$

$$\left[c-2a-(b-a)x\right]F+a(1-x)F(a+1)-(c-a)F(a-1)=0, \qquad (1.87)$$

$$c\left[a-(c-b)x\right]F-ac(1-x)F(a+1)+(c-a)(c-b)xF(c+1)=0, \qquad (1.88)$$

$$\left[a-1-(c-b-1)x\right]F-(c-1)(1-x)F(c-1)+(c-a)F(a-1)=0, \qquad (1.89)$$

$$\left[c-2b+(b-a)x\right]F+b(1-x)F(b+1)-(c-b)F(b-1)=0, \qquad (1.90)$$

$$c\left[b-(c-a)x\right]F-bc(1-x)F(b+1)+(c-a)(c-b)xF(c+1)=0, \qquad (1.91)$$

$$\left[b-1-(c-a-1)x\right]F-(c-1)(1-x)F(c-1)+(c-b)F(b-1)=0, \qquad (1.92)$$

$$c\left[c-1-(2c-a-b-1)x\right]F-c(c-1)(1-x)F(c-1)+$$

$$+(c-a)(c-b)xF(c+1)=0, \qquad (1.93)$$

(1.79) - (1.93) chiziqli bog'liqliklarning to`g'riligini isbotlashda (1.42) qatordan foydalaniladi. Misol tariqasida (1.79) tenglikni to`g'riligini isbotlaymiz.

(1.41) va (1.42) yordamida (1.79) tenglikni quyidagicha yozib olamiz:

$$(b-a)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}x^n=b\sum_{n=0}^{\infty}\frac{(a)_n(b+1)_n}{n!(c)_n}x^n-a\sum_{n=0}^{\infty}\frac{(a+1)_n(b)_n}{n!(c)_n}x^n=$$

$$=b\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+1+n)\Gamma(c)}{\Gamma(a)\Gamma(b+1)\Gamma(c+n)\,n!}x^n-a\sum_{n=0}^{\infty}\frac{\Gamma(a+1+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a+1)\Gamma(b)\Gamma(c+n)\,n!}x^n.$$

Bundan va (1.6), (1.41) formulalardan foydalanib, oxirgi tenglikni ushbu

$$(b-a)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}x^n=$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!}\left[(b+n)-(a+n)\right]x^n=(b-a)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}x^n.$$

ko`rinishda yozib olamiz. Bu tenglikni o`ng va chap tomonlari teng. Demak, (1.79) tenglik o`rinlidir.

Agar yondosh gipergeometrik funksiyalarda ikkita parametr bir xil bo`lsa, u holda ular o`rtasida ushbu munosabatlar o`rinlidir:

$$(c-a)F(a-1,b,c;x)+(2a-c-ax+bx)F(a,b,c;x)+$$
$$+a(x-1)F(a+1,b,c;x)=0, \qquad (1.94)$$

$$(c-b)F(a,b-1,c;x)+(2b-c-bx+ax)F(a,b,c;x)+$$
$$+b(x-1)F(a,b+1,c;x)=0, \qquad (1.95)$$

$$c(c-1)(x-1)F(a,b,c-1;x)+c\left[c-1-(2c-a-b-1)x\right]F(a,b,c;x)+$$
$$+(c-a)(c-b)x\,F(a,b,c+1;x)=0. \qquad (1.96)$$

### 1. 2. 5. Gipergeometrik funksiyalarni differensiallash qoidalari

$F(a,b,c;x)$ gipergeometrik funksiya uchun quyidagi differensiallash qoidalari o`rinli[23: 110-111 betlar]:

$$\frac{d^n}{dx^n}F(a,b,c;x)=\frac{(a)_n(b)_n}{(c)_n}\,F(a+n,b+n,c+n;x), \qquad (1.97)$$

$$\frac{d^n}{dx^n}\left[x^{a+n-1}F(a,b,c;x)\right]=(a)_n\,x^{a-1}F(a+n,b,c;x), \qquad (1.98)$$

$$\frac{d^n}{dx^n}\left[x^{c-1}F(a,b,c;x)\right]=(c-n)_n\,x^{c-n-1}F(a,b,c-n;x), \qquad (1.99)$$

$$\frac{d^n}{dx^n}\left[x^{c-a+n-1}(1-x)^{a+b-c}F(a,b,c;x)\right]=$$
$$=(c-n)_n\,x^{c-a-1}(1-x)^{a+b-c-n}F(a-n,b,c;x), \qquad (1.100)$$

$$\frac{d^n}{dx^n}\left[(1-x)^{a+b-c}F(a,b,c;x)\right]=\frac{(c-a)_n(c-b)_n}{(c)_n}\,(1-x)^{a+b-c-n}F(a,b,c+n;x), \qquad (1.101)$$

$$\frac{d^n}{dx^n}\left[(1-x)^{a+n-1}F(a,b,c;x)\right]=\frac{(-1)^n(a)_n(c-b)_n}{(c)_n}\,(1-x)^{a-1}F(a+n,b,c+n;x), \qquad (1.102)$$

$$\frac{d^n}{dx^n}\left[x^{c-1}(1-x)^{b-c+n}F(a,b,c;x)\right]=$$

$$=(c-n)_n x^{c-1-n}(1-x)^{b-c}F(a-n,b,c-n;x). \qquad (1.103)$$

(1.97) - (1.103) formulalarning to`g'riligini matematik induksiya usuli yordamida isbotlash mumkin. Misol tariqasida (1.97) formulani to`g'riligini isbotlaymiz.

Agar (1.97) formulada $n=1$ bo`lsa, u holda quyidagi tenglikka ega bo`lamiz:

$$\frac{d}{dx}F(a,b,c;x)=\frac{ab}{c}\,F(a+1,b+1,c+1;x).$$

Haqiqatan ham, (1.42) qatorga va (1.41), (1.6) formulalarga ko`ra ushbu

$$\frac{d}{dx}F(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}n x^{n-1}=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!}n x^{n-1}=$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+n)n!}\frac{ab}{c}n x^{n-1}=\{k=n-1,\ (k+1)!=k!(k+1)\}=$$

$$=\frac{ab}{c}\sum_{k=0}^{\infty}\frac{\Gamma(a+k+1)\Gamma(b+k+1)\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+k+1)k!}x^k=\frac{ab}{c}\sum_{k=0}^{\infty}\frac{(a+1)_k(b+1)_k}{k!(c+1)_k}x^k=$$

$$=\frac{ab}{c}F(a+1,b+1,c+1;x)$$

tenglik o`rinli bo`ladi.

Shunday qilib, $n=1$ da (1.97) formula to`g'ri. Faraz qilaylik bu formula $n=r$ da ham to`g'ri bo`lsin, u holda $n=r+1$ to`g'riligini isbotlaymiz:

$$\frac{d^{r+1}}{dx^{r+1}}F(a,b,c;x)=\frac{d}{dx}\left[\frac{d^r}{dx^r}F(a,b,c;x)\right]==\frac{(a)_r(b)_r}{(c)_r}\times$$

$$\times\frac{d}{dx}\left[F(a+r,b+r,c+r;x)\right]=\frac{(a)_r(b)_r}{(c)_r}\frac{(a+r)(b+r)}{(c+r)}F(a+r+1,b+r+1,c+r+1;x)=$$

$$=\frac{(a)_{r+1}(b)_{r+1}}{(c)_{r+1}}F(a+r+1,b+r+1,c+r+1;x).$$

Shunday qilib, (1.97) formula $n=r+1$ uchun ham to`g'ri. Matematik induksiya usuliga ko`ra (1.97) tenglik o`rinlidir. Qolgan differensiallash formulalari ham shu usulda isbotlanadi.

## I bob bo‘yicha xulosalar

Bizga ma’lumki, ko‘p o‘zgaruvchili gipergeometrik funksiyalarni tadqiq etishda maxsus funksiyalar muhim ahamiyat kasb etadi, shu sababali mazkur bobda gamma va beta funksiyalar, Gaussning gipergeometrik funksiyasi haqida tushunchalar, Gauss tenglamasining Kimmer yechimlari, ularning integral ifodalari, gipergeometrik funksiylarni analitik davom ettirish va ular orasidagi asosiy funksional bog’lanishlar haqida so‘z yuritilgan. Shu bilan birga gipergeometrik funksiyalarni differensiallash qoidalari keltirilgan.

## II BOB. IKKI O‘ZGARUVCHILI GIPERGEOMETRIK FUNKSIYALAR VA DIFFERENSIAL OPERATORLAR

### 2. 1- §. Ikki o`zgaruvchili gipergeometrik funksiyalar haqida tushunchalar

**I.** Ushbu ikkita

$$x(1-x)z_{xx}+(1-x)y\,z_{xy}+\left(c-(a+b+1)x\right)z_x-byz_y-abz=0,\quad(2.1)$$

$$y(1-y)z_{yy}+(1-y)x\,z_{xy}+\left(c-(a+b'+1)y\right)z_y-b'xz_x-ab'z=0,\quad(2.2)$$

ikkinchi tartibli xususiy hosilali differensial tenglamalarni qaraymiz, bu yerda $z=z(x,y)-$noma’lum funksiya, $c,\ a,\ b,\ b'-$ parametrlar ixtiyoriy haqiqiy yoki kompleks sonlar bo`lishi mumkin.

**1. 1 -Ta’rif.** Ikki o`zgaruvchili $F_1\left(a,\,b,\,b',\,c;\,x,y\right)$ gipergeo-metrik funksiya deb, quyidagi

$$F_1\left(a,\,b,\,b',\,c;\,x,y\right)=\sum_{n,m=0}^{\infty}\frac{(a)_{n+m}(b)_n(b')_m}{(c)_{n+m}n!m!}\,x^ny^m\qquad(2.3)$$

qator bilan aniqlanuvchi funksiyaga aytiladi.

(2.3) funksiya (2.1) va (2.2) tenglamalarning yechimi bo`ladi. Buni o`rniga qo`yish usuli yordamida ko`rsatish mumkin. (2.2) qator $|x|<1,\ |y|<1$ sohada absolyut va tekis yaqinlashuvchi bo`ladi.

$F_1\left(a,b,b',\,c;\,x,y\right)$ gipergeometrik funksiyaning integral ifodasi quyidagi

$$F_1\left(a,b,b',\,c;\,x,y\right)==\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b(1-yt)^{b'}}dt,\quad\left.\begin{matrix}\mathrm{Re}\,a>0,\\ \mathrm{Re}(c-a)>0\end{matrix}\right\}\qquad(2.4)$$

ko`rinishda bo`ladi. (1.107) integral ifodaga **Pikar integrali** deyiladi.

$F_1\left(a,b,b',\,c;\,x,y\right)$**gipergeometrik funksiya quyidagi xossalarga ega:**

**1. Ushbu tengliklar o`rinli:**

$$F_1\left(a,b,b',\,c;\,x,y\right)=F_1\left(a,b',b,\,c;\,x,y\right),\qquad(2.5)$$

$$F_1\left(a,b,b',\,c;\,x,x\right)=F\left(a,\,b'+b,\,c;\,x\right),\qquad(2.6)$$

$$F_1\left(a,b,b',\,c;\,x,1\right)=\frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}F\left(a,b,\ c-b';\,x\right),\quad \mathrm{Re}(c-a-b')>0,\quad(2.7)$$

$$F_1\left(a,b,b',\,c;1,y\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F\left(a,b',\ c-b;\,y\right),\quad \mathrm{Re}(c-a-b)>0,\quad(2.8)$$

$$F_1\left(a,b,b',\,c;\,0,y\right)=F_1\left(a,0,b',\,c;\,x,y\right)=F\left(a,\ b',\,c;\,y\right),\qquad(2.9)$$

$$F_1\left(a,b',b,\,c;\,x,0\right)=F_1\left(a,b,0,\,c;\,x,y\right)=F\left(a,\ b,\,c;\,x\right),\qquad(2.10)$$

$$F_1\left(0,b,b',\,c;\,x,0\right)=F_1\left(a,0,b',\,c;\,x,0\right)=F_1\left(a,b,0,\,c;\,x,0\right)=$$
$$=F_1\left(a,b,b',\,c;\,0,0\right)=F_1\left(0,b,b',\,c;\,0,y\right)=F_1\left(a,b,0,\,c;\,0,y\right)=1\ .\qquad(2.11)$$

**2. Quyidagi rekurrent munosabatlar o`rinli:**

$$(c-b-b'-1)F_1(a, b, b', c; x, y)+bF_1(a, b+1, b', c; x, y)+$$
$$+b'F_1(a, b, b'+1, c; x, y)=(c-1)F_1(a, b, b', c-1; x, y), \quad (2.12)$$

$$(c-a-1)F_1(a, b, b', c; x, y)+aF_1(a+1, b, b', c; x, y)+$$
$$=(c-1)F_1(a, b, b', c-1; x, y). \quad (2.13)$$

**3.** (2.3) qatorni $x$ va $y$ bo`yicha hadma-ham differensiallab, quyidagi **differensiallash formulalarini** hosil qilamiz:

$$\frac{\partial}{\partial x}F_1(a, b, b', c; x, y)=\frac{ab}{c}F_1(a+1, b+1, b', c+1; x, y), \quad (2.14)$$

$$\frac{\partial}{\partial y}F_1(a, b, b', c; x, y)=\frac{ab'}{c}F_1(a+1, b, b'+1, c+1; x, y), \quad (2.15)$$

$$\frac{x}{b}\frac{\partial}{\partial x}F_1(a, b, b', c; x, y)=F_1(a, b+1, b', c; x, y)-F_1(a, b, b', c; x, y), \quad (2.16)$$

$$\frac{y}{b'}\frac{\partial}{\partial y}F_1(a, b, b', c; x, y)=F_1(a, b, b'+1, c; x, y)-F_1(a, b, b', c; x, y). \quad (2.17)$$

**4. Ushbu tengliklar o`rinli:**

$$F_1(a, b, b', b+b'; x, y)=(1-y)^{-a}F(a, b, b+b'; (x-y)/(1-y)), \quad (2.18)$$

$$F_1(a, b, b', c; x, y)=(1-x)^{-b}(1-y)^{-b'}F_1\left(c-a, b, b', c; \frac{x}{x-1}, \frac{y}{y-1}\right), \quad (2.19)$$

$$F_1(a, b, b', c; x, y)=(1-x)^{-a}F_1\left(a, c-b-b', b', c; \frac{x}{x-1}, \frac{y-x}{1-x}\right), \quad (2.20)$$

$$F_1(a, b, b', c; x, y)=(1-y)^{-a}F_1\left(a, c-b-b', b', c; \frac{y-x}{y-1}, \frac{y}{y-1}\right), \quad (2.21)$$

$$F_1(a, b, b', c; x, y)=(1-x)^{c-a-b}(1-y)^{-b'}F_1\left(c-a, c-b-b', b' c; x, \frac{x-y}{1-y}\right), \quad (2.22)$$

$$F_1(a, b, b', c; x, y)=(1-x)^{-b}(1-y)^{c-a-b'}F_1\left(c-a, b, c-b-b', c; \frac{x-y}{x-1}, y\right). \quad (2.23)$$

**5.** $F_1(a, b, b', c; x, y)$ **gipergeometrik funksiyaning ikki karrali Eyler tipidagi integral orqali ifodasi**[23: 224 bet]**:**

$$F_1(a, b, b', c; x, y)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\iint\limits_{\substack{u\geq 0,\ \upsilon\geq 0\\ u+\upsilon\leq 1}}\frac{u^{b-1}\upsilon^{b'-1}}{(1-u-\upsilon)^{1+b+b'-c}}\times$$
$$\times(1-ux-\upsilon y)^{-a}du d\upsilon, \quad \mathrm{Re}\, b>0, \quad \mathrm{Re}\, b'>0, \quad \mathrm{Re}(c-b-b')>0. \quad (2.24)$$

**II.** Ushbu ikkita

$$x(1-x)z_{xx}-xy\,z_{xy}+(\gamma-(\alpha+\beta+1)x)z_x-\beta y z_y-\alpha\beta z=0 \quad (2.25)$$

$$y(1-y)z_{yy}-x\,yz_{xy}+(\gamma'-(\alpha+\beta'+1)y)z_y-\beta' x z_x-\alpha\beta' z=0 \quad (2.26)$$

ikkinchi tartibli xususiy hosilali differensial tenglamalarni qaraymiz, bu yerda $z = z(x, y) -$ noma'lum funksiya, $\alpha, \beta, \beta', \gamma, \gamma'$ parametrlar ixtiyoriy haqiqiy yoki kompleks sonlar bo`lishi mumkin.

1.2-**Ta'rif.** Ikki o`zgaruvchili $F_2(\alpha, \beta, \beta', \gamma; \gamma' \; x, y)$ gipergeometrik funksiya deb quyidagi

$$F_2(\alpha, \beta, \beta', \gamma; \gamma' \; x, y) = \sum_{n,m=0}^{\infty} \frac{(\alpha)_{n+m}(\beta)_n(\beta')_m}{(\gamma)_n(\gamma')_m n! m!} x^n y^m \qquad (2.27)$$

qator bilan aniqlanuvchi funksiyaga aytiladi.

(2.27) funksiya (2.25) va (2.24) tenglamalarning yechimi bo`ladi. Buni o`rniga qo`yish usuli yordamida ko`rsatish mumkin. (2.27) qator $|x| + |y| < 1$ sohada absolyut va tekis yaqinlashuvchi bo`ladi.

$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$ **gipergeometrik funksiya quyidagi xossalarga ega:**

**$1^0$. Ushbu tengliklar o`rinli:**

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = F_2(\alpha, \beta', \beta, \gamma', \gamma : y, x), \qquad (2.28)$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, 0) = F_2(\alpha, \beta, 0, \gamma, \gamma'; x, y) = F(\alpha, \beta, \gamma; x), \qquad (2.29)$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; 0, y) = F_2(\alpha, 0, \beta', \gamma, \gamma'; x, y) = F(\alpha, \beta', \gamma'; y), \qquad (2.30)$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; 0, 0) = 1, \qquad (2.31)$$

$$F_2(\alpha, \beta, \beta', \beta, \gamma'; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta', \gamma'; \frac{y}{1-x}\right), \qquad (2.32)$$

$$F_2(\alpha, \beta, \beta', \alpha, \alpha; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} F\left(\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right), \qquad (2.33)$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\alpha} F_2\left(\alpha, \gamma-\beta, \beta', \gamma, \gamma', \frac{x}{x-1}, \frac{y}{1-x}\right). \qquad (2.34)$$

**$2^0$. Quyidagi rekurrent munosabatlar o`rinli:**

$$\alpha F_2(\alpha+1, \beta, \beta', \gamma, \gamma'; x, y) - \beta F_2(\alpha, 1+\beta, \beta', \gamma, \gamma'; x, y) -$$

$$-\beta' F_2(\alpha, \beta, 1+\beta', \gamma, \gamma'; x, y) = (\alpha - \beta - \beta') F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y), \qquad (2.35)$$

$$\frac{\beta}{\gamma} x F_2(1+\alpha, 1+\beta, \gamma+1, \gamma'; x, y) + \frac{\beta'}{\gamma'} y F_2(1+\alpha, \beta, 1+\beta,' \gamma, \gamma'+1; x, y) =$$

$$= F_2(1+\alpha, \beta, \beta', \gamma, \gamma'; x, y) - F_2(\alpha, \beta, \beta' \gamma, \gamma'; x, y). \qquad (2.36)$$

**$3^0$.(2.27)qatorni $x$ va $y$ bo`yicha hadma–ham differensialab, quyidagi differensiallash formulalarini hosil qilamiz:**

$$\frac{\partial}{\partial x} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \frac{\alpha\beta}{\gamma} F_2(\alpha+1, \beta+1, \beta', \gamma+1, \gamma'; x, y), \qquad (2.37)$$

$$\frac{\partial}{\partial y} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \frac{\alpha\beta'}{\gamma'} F_2(\alpha+1, \beta, \beta'+1, \gamma, \gamma'+1; x, y), \qquad (2.38)$$

$$\frac{\partial}{\partial x} x^{\gamma-1} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = (\gamma-1) x^{\gamma-2} F_2(\alpha, \beta, \beta', \gamma-1, \gamma'; x, y). \qquad (2.39)$$

$$\frac{\partial}{\partial y} y^{\gamma'-1} F_2\left(\alpha,\beta,\beta',\gamma,\gamma';x,y\right)=\left(\gamma'-1\right)y^{\gamma'-2}F_2\left(\alpha,\beta,\beta',\gamma,\gamma'-1;x,y\right). \quad (2.40)$$

**4⁰.** $F_2(\alpha,\beta,\beta',\gamma;\gamma' x,y)$ **gipergeometrik funksiyaning Gauss funksiyasi orqali ifodasi:**

$$F_2\left(\alpha,\beta,\beta',\gamma,\gamma';x,y\right)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\beta')_k}{(\gamma)_k(\gamma')_k k!}x^k y^k \times$$
$$\times F(\alpha+k,\beta+k,\gamma+k;x)\cdot F\left(\alpha+k,\beta'+k,\gamma'+k;y\right). \quad (2.41)$$

**III.** Ushbu ikkita

$$x(1-x)z_{xx}-xy\,z_{xy}+\left(\gamma-(\alpha+\beta+1)x\right)z_x-\beta yz_y-\alpha\beta z=0 \quad (2.42)$$
$$y(1-y)z_{yy}-x\,yz_{xy}+\left(\gamma'-(\alpha+\beta'+1)y\right)z_y-\beta'xz_x-\alpha\beta'z=0 \quad (2.43)$$

ikkinchi tartibli xususiy hosilali differensial tenglamalarni qaraymiz, bu yerda $z=z(x,y)-$noma'lum funksiya, $\alpha,\beta,\beta',\gamma,\gamma'$ parametrlar ixtiyoriy haqiqiy yoki kompleks sonlar bo`lishi mumkin.

**1.3-Ta'rif.** Ikki o`zgaruvchili $F_3(\alpha,\alpha',\beta,\beta',\gamma,;x,y)$ Gorn gipergeometrik funksiyasi deb quyidagi

$$F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n \quad (2.44)$$

qator bilan aniqlanuvchi funksiyaga aytiladi.

(2.33) funksiya (2.42) va (2.43) tenglamalarning yechimi bo`ladi. Buni o`rniga qo`yish usuli yordamida tekshirish mumkin. (2.44) qator $|x|<1,\ |y|<1$ sohada absolyut va tekis yaqinlashuvchi bo`ladi.

$F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right)$ **funksiyaning ayrim xossalari:**

**1⁰.** $F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right)=F_3\left(\alpha',\alpha,\beta',\beta,\gamma;x,y\right);$ (2.45)

**2⁰.** $F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,0\right)=F_3\left(\alpha,0,\beta,\beta',\gamma;x,y\right)=$
$=F_3\left(\alpha,\alpha',\beta,0,\gamma;x,y\right)=F(\alpha,\beta,\gamma;x);$ (2.46)

**3⁰.** $F_3\left(\alpha,\alpha',\beta,\beta',\gamma;0,y\right)=F_3\left(0,\alpha',\beta,\beta',\gamma;x,y\right)=$
$=F_3\left(\alpha,\alpha',0,\beta',\gamma;x,y\right)=F\left(\alpha',\beta',\gamma;y\right);$ (2.47)

**4⁰.** $F_3\left(\alpha,\alpha',\beta,\beta',\gamma;0,0\right)=1$ ; (2.48)

**5⁰.** $\alpha F_3\left(\alpha+1,\alpha',\beta,\beta',\gamma;x,y\right)-\beta F_3\left(\alpha,\alpha',1+\beta,\beta',\gamma;x,y\right)=$
$=(\alpha-\beta)F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right)$ ; (2.49)

**6⁰.** $\alpha' F_3\left(\alpha,\alpha'+1,\beta,\beta',\gamma;x,y\right)-\beta' F_3\left(\alpha,\alpha',\beta,\beta'+1,\gamma;x,y\right)=$
$=\left(\alpha'-\beta'\right)F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right);$ (2.50)

**7⁰.** $\frac{\beta}{\gamma}x\,F_3\left(1+\alpha,\alpha',1+\beta,\beta',1+\gamma;x,y\right)=$
$=F_3\left(1+\alpha,\alpha',\beta,\beta',\gamma;x,y\right)-F_3\left(\alpha,\alpha',\beta,\beta',\gamma;x,y\right)$ ; (2.51)

**8⁰.** $$\frac{\beta'}{\gamma} y F_3(\alpha,\alpha'+1,\beta,\beta'+1,\gamma+1;x,y)=$$
$$=F_3(\alpha,\alpha'+1,\beta,\beta',\gamma;x,y)-F_3(\alpha,\alpha',\beta,\beta,'\gamma;x,y)\ ; \qquad (2.52)$$

**9⁰.** $$\frac{\partial}{\partial x}F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\frac{\alpha\beta}{\gamma}F_3(\alpha+1,\alpha',\beta+1,\ \beta',\gamma+1;x,y)\ ; \qquad (2.53)$$

**10⁰.** $$\frac{\partial}{\partial y}F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\frac{\alpha'\beta'}{\gamma}F_3(\alpha,\alpha'+1,\beta,\beta'+1,\gamma+1;x,y); \qquad (2.54)$$

**11⁰.** $$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=$$
$$=(-y)^{\alpha'}\frac{\Gamma(\gamma)\Gamma(\beta'-\alpha')}{\Gamma(\beta')\Gamma(\gamma-\alpha')}H_2\left(1-\gamma+\alpha',\alpha',\alpha,\beta,1-\beta'+\alpha';\frac{1}{y};\ -x\right)+$$
$$+(-y)^{\beta'}\frac{\Gamma(\gamma)\Gamma(\alpha'-\beta')}{\Gamma(\alpha')\Gamma(\gamma-\beta')}H_2\left(1-\gamma+\beta',\beta',\alpha,\beta,1-\alpha'+\beta';\frac{1}{y};-x\right), \qquad (2.55)$$

bu yerda $H_2(\alpha,\beta,\gamma,\delta;\varepsilon;x,y)$ - Gorn gipergeometrik funksiyasi [23]:

a) $$H_2(\alpha,\beta,\gamma,\delta;\varepsilon;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m m!n!}x^m y^n\ , \qquad (2.56)$$

b) $$H_2(\alpha,\beta,\gamma,\delta;\varepsilon;x,y)=\frac{\Gamma(\varepsilon)}{\Gamma(\beta)\Gamma(\varepsilon-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\varepsilon-\beta-1}\cdot(1-tx)^{-\alpha}\times$$
$$\times F\,(\gamma,\delta,1-\alpha;\ -y(1-tx))\,dt,\ [\mathrm{Re}\,\varepsilon>0,\ \mathrm{Re}\,\beta>0,\mathrm{Re}(\varepsilon-\beta)>0], \qquad (2.57)$$

c) $$H_2(\alpha,\beta,\gamma,\delta,\varepsilon;0,y)=F(\gamma,\delta,1-\alpha;\ -y). \qquad (2.58)$$

**IV. Gauss – Ostrogradskiy formulasi** [30], [26: 17 bet]:
$$\int_\Omega\sum_{i=1}^{n}\frac{\partial P_i(x)}{\partial x_i}d\Omega=\int_S\sum_{i=1}^{n}P_i(x)\cos(\nu,x_i)ds, \qquad (2.59)$$

bu yerda $P_i(x)=P_i(x_1,x_2,\dots,x_n)$, $S$ – $\Omega$ sohasining chegarasi bo`lib, u bo`lakli silliq sirt, $\nu_i=\cos(\nu,x_i)$ esa $S$ sirtga o`tkazilgan tashqi $\nu=(\nu_1,\dots,\nu_n)$ normalning yo`naltiruvchi kosinuslari.

Agar $P_1=P(x,y),\ P_2=Q(x,y),$ ikki o`zgaruvchili funksiya bo`lsa, u holda quyidagi
$$\iint_\Omega\left[\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right]dxdy=\int_{\partial\Omega}Qdx+Pdy \qquad (2.60)$$
Grin formulasi o`rinli, $\partial\Omega-\Omega$ sohasining chegarasi.

## 2.2- §. Kasr tartibli integro-differensial operatorlar haqida tushunchalar

### 2. 2. 1. Kasr tartibli integral va differensial operatorlarning ta’riflari. Uning ayrim sodda xossalari

**2.2-Ta'rif [48:** 2-§. G l. 1**].** Agar $f(x)\in L_1(a,b)$ bo`lsa, u holda

$$D_{ax}^{-\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}},\qquad x>a\ , \tag{2.61}$$

$$D_{xb}^{-\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}},\qquad x<b \tag{2.62}$$

integrallarga kasr tartibli(tartibi $\alpha$) **integral** deyiladi, bu yerda $\alpha>0$.

$f(x)\in L_1(a,b)-$ degani $\int_a^b |f(x)|dx<\infty$ mavjud.

**1.5-Ta'rif.** Agar $0<\alpha<\infty$ bo`lib, $f(x)\in L_1(a,b)$ va $\frac{d^{n-1}}{dx^{n-1}} D_{ax}^{-(n-\alpha)} f(x)$ funksiya $(a,b)$ oraliqning deyarli hamma joyida hosilaga ega ($(a,b)$da jamlanuvchi bo`lishi shart emas) bo`lsa, u holda

$$D_{ax}^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}, \tag{2.63}$$

$$D_{xb}^{\alpha} f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)\,dt}{(t-x)^{\alpha-n+1}} \tag{2.64}$$

funksiyalar kasr tartibli (tartibi $\alpha$) **hosila** (differensial) deyiladi, bu yerda $n=[\alpha]+1,$ ya'ni $n\geq 1$ bo`lsa, $n-1\leq\alpha\leq n,$ $[\alpha]-\alpha$ sonining butun qismi.

Agar $\alpha=n\geq 1$ bo`lsa, u holda

$$D_{ax}^{\alpha} f(x)=\frac{d^{\alpha} f(x)}{dx^{\alpha}}, \tag{2.65}$$

$$D_{xb}^{\alpha} f(x)=(-1)^{\alpha}\frac{d^{\alpha} f(x)}{dx^{\alpha}}. \tag{2.66}$$

oddiy differensialni beradi.

**Quyidagi belgilashlarni kiritamiz.** $M=[a,b],\ -\infty\leq a<b\leq+\infty,$ deb chekli kesma, yarim o`q yoki to`liq o`qni belgilaymiz. $M$ da aniqlagan va Lebeg ma'nosida jamlanuvchi funksiyalar fazosini $L_p(M)$ bilan belgilaymiz. Bu fazoning normasini quyidagicha

$$\| f \|_{L_p(M)} = \left\{ \int_M | f(x)|^p \, dx \right\}^{1/p}, \qquad 1 \le p < \infty \qquad (2.67)$$

kiritamiz. Keyinchalik $L_p = L_p(M)$, $\| f \|_p = \| f \|_{L_p(M)}$ belgilashlardan foydalanamiz.

$M$ sohada cheksiz marta differensiallanuvchi finit funksiyalar sinfini $C_0^\infty = C_0^\infty(M)$ bilan belgilaymiz. $f(x)$ funksiya $M = [a, b]$ kesmada finit deyiladi, agar kesmaning chetki nuqtalari atrofida $f(x) = 0$ bo`lsa. $C_0^\infty = C_0^\infty(M)$ sinf $L_p(M)$, $1 \le p < \infty$ da to`ladir.

Agar $M = [a, b]$ chekli kesma bo`lsa, u holda hamma ko`phadlar to`plami $L_p(M)$, $1 \le p < \infty$ da to`la [24], [25].

**1.6 -Ta'rif.** Agar ixtiyoriy $x_1, x_2 \in (a,b)$ uchun ushbu

$$| f(x_1) - f(x_2)| \le K |x_1 - x_2|^\lambda \qquad (2.68)$$

tengsizlik bajarilsa, u holda $(a,b)$ intervalda $f(x)$ funksiya $\lambda \in (0,1]$ darajali Gyolder shartini qanoatlantiradi deyiladi, bu yerda $K -$ Gyolder o`zgarmasi, $\lambda -$ Gyolder darajasi.

$H^\lambda(M)$ deb, $(a,b)$ intervalda $\lambda \in (0,1]$ darajali Gyolder shartini qanoatlantiruvchi barcha funksiyalar sinfini belgilaymiz.

**1.7-Ta'rif.** Agar (1.171) tengsizlikda $\lambda = 1$ bo`lib, quyidagi

$$| f(x_1) - f(x_2)| \le K |x_1 - x_2| \qquad (2.69)$$

tengsizlik bajarilsa, u holda $(a,b)$ intervalda $f(x)$ funksiya Lipshits shartini qanoatlantiradi deyiladi.

$H^1([a,b])$ deb, $(a,b)$ intervalda Lipshits shartini qanoatlantiruvchi barcha funksiyalar sinfini belgilaymiz.

**1.8-Ta'rif.** Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ mavjud bo`lsaki, soni chekli va har ikkisi o`zaro kesishmaydigan har qanday

$\left[a_k,b_k\right]\subset M,\ k=1,2,\dots,n$ kesmalar sistemasi uchun $\bigcup_{k=1}^{n}\left[a_k,b_k\right]\subset\left[a,b\right]$, $\sum_{k=1}^{n}(b_k-a_k)<\delta$ shartlar bajarilganda $\sum_{k=1}^{n}\left|f(b_k)-f(a_k)\right|<\varepsilon$ tengsizlik o`rinli bo`lsa, u holda $f(x)$ funksiya $\left[a,b\right]$ kesmada absolyut uzluksiz deyiladi.

Hamma absolyut uzluksiz funksiyalar sinfini $AC(M)$ deb belgilaymiz.

**Absolyut uzluksiz funksiya uchun ushbu tushuncha o`rinlidir:** $\left[a,b\right]$ kesmada aniqlangan absolyut uzluksiz $f(x)$ funksiyaning hosilasi $f'(x)=\varphi(x)$ jamlanuvchi va har bir $x$ uchun $f(x)=f(a)+\int_a^x\varphi(t)dt$ bo`ladi, ya'ni

$$f(x)\in AC(M)\ \Leftrightarrow\ f(x)=f(a)+\int_a^x\varphi(t)dt,\quad \int_a^x\varphi(t)dt<\infty.$$

Bundan ixtiyoriy absolyut uzluksiz funksiyalar $M=\left[a,b\right]$ kesmaning deyarli hamma joyida jamlanuvchi $f'(x)$ hosilaga ega ekanligi kelib chiqadi(teskarisi o`rinli emas).

$H^{\lambda}(M)$ va $H^{1}(M)$ shartlarni qanoatlantiruvchi funksiyalarning xossalari:

1) Agar $(a,b)$ intervalda $f(x)$ funksiya Lipshits shartini qanoatlantirsa, u holda $f(x)$ funksiya $(a,b)$ intervalda $\left|f'(x)\right|<K$ chekli hosilaga ega;

2) Agar $f(x)$ funksiya $(a,b)$ chekli intervalda $\lambda$ ko`rsatkich bilan Gyolder shartini qanoatlantirsa, u holda bu funksiya $\delta<\lambda$ ko`rsatkich bilan ham Gyolder shartini qanoatlantiradi;

3) Agar $f_1(x)$ va $f_2(x)$ $(a,b)$ intervalda mos ravishda $H^{\lambda_1}(M)$ va $H^{\lambda_2}(M)$ shartlarni qanoatlantirsa, u holda $f_1(x)+f_1(x)$, $f_1(x)f_2(x)$, $f_1(x)/f_2(x)$, $\left(f_2(x)\neq 0\right)$ funksiyalar $\lambda<\min(\lambda_1,\lambda_2)$ daraja bilan Gyolder shartini qanoatlantiradi;

4) $AC(M)$ absolyut uzluksiz funksiyalar sinfi $H^1(M)$ sinfga nisbatan kengroq sinfdir:

5) Quyidagi $H^1(M)\subset AC(M)$ munosabat o`rinlidir. Teskari munosabat o`rinli emas, ya'ni $H^1(M)\supseteq AC(M)$, haqiqatan ham, $f(x)=(x-a)^{\alpha}\in AC(M)$, lekin $0<\alpha<1$ bo`lganda $(x-a)^{\alpha}\notin H^1(M)$, chunki $x=a$ da (2.69) shart bajarilmaydi.

**1.9-Ta'rif.** $\rho(x)$ manfiy bo`lmagan funksiya bo`lsin. Agar $\rho(x)f(x)\in H^{\lambda}(M)$ bo`lsa, u holda $f(x)\in H^{\lambda}(\rho)\equiv H^{\lambda}(M;\rho)$ tegishli bo`ladi, bu yerda $\rho(x)=\prod_{k=1}^{n}|x-x_k|^{\mu_k}$, $\mu_k-$ haqiqiy sonlar, $x_k\in M$ bo`lishi mumkin.

Xuddi shunday $\rho(x)$ vaznli jamlanuvchi funksiyalar sinfini $L_p(\rho)\equiv L_p(M;\rho)=\{f(x):\ \rho(x)f(x)\in L_p(M)\}$ belgilaymiz.

**1.10-Ta'rif.** Agar $f(x)\in H^{\lambda}(\rho)$ bo`lib, $f(x)=\frac{f_0(x)}{\rho(x)}$, $f_0(x_k)=0$ va $f_0(\infty)=0,\ f_0(x)\in H^{\lambda}(M)$ shartlar bajarilsa, u holda $f(x)\in H_0^{\lambda}(\rho)\equiv H_0^{\lambda}(M;\rho)$ bo`ladi.

**1.11-Ta'rif.** $H_0^{\lambda}(M)$ deb, $f(x)$ funksiya $H^{\lambda}(M)$ sinfga tegishli bo`lib, $f(a)=0$ va $f(b)=0$ shartlarni qanoatlantiruvchi barcha funksiyalar sinfini belgilaymiz.

Quyidagi $C^1(M)\in H_0^{\lambda}(M;\rho)\in L_1(M),\quad M=[a,b]$ munosabat o`rinli bo`lib, ushbu

$$\alpha\|f\|_{L_1}\le\|f\|_{H_0^{\lambda}}\le\beta\|f\|_{C^1}$$

tengsizlik bajariladi, bu yerda $\lambda\le\mu_k\le\lambda+1,\quad k=1,2,3,\ldots,n;\ C^m(M)-\ m$ marta uzluksiz differensiallanuvchi funksiyalar sinfi bo`lib, ushbu $\|f\|_{C^m}=\max_{x\in M}\sum_{k=1}^{m}\left|f^{(k)}(x)\right|,\quad m=0,1,2,\ldots$ normaga ega.

**Kasr tartibli integral operatorlar uchun quyidagi teoremalar o`rinli.**

**1.1 - Teorema**[49: 19 bet],[48]. Agar $k$ va $\lambda$ sonlari uchun $k \geq 0$, $\lambda > 0$, $k+\lambda<1$ shartlar bajarilgan bo`lib, $f(x)$ funksiya $(a,b)$ intervalda $k$ ko`rsatkich bilan Gyolder shartini qanoatlantirsa va kichik $x-a$ lar uchun $f(x)=O\left((x-a)^k\right)$ bahoga ega bo`lsa, u holda $D_{ax}^{-\lambda} f(x)$ funksiya $(a,b)$ intervalda $k+\lambda$ ko`rsatkich bilan Gyolder shartini qanoatlantiradi va kichik $x-a$ lar uchun

$$D_{ax}^{-\lambda} f(x)=O\left((x-a)^{k+\lambda}\right) \qquad (2.70)$$

baho o`rinlidir.

1.1 teoremani to`liq isboti [48: 3-§. Gl. 1] da keltirilgan.

**1. 2 –Teorema.** Agar $p>1$, $\frac{1}{p}<\alpha<\frac{1}{p}+1$ yoki $p=1$, $1\leq\alpha<2$ bo`lib, $f(x)\in L_p(a,b)$ bo`lsa, u holda $D_{ax}^{-\alpha} f(x)$ funksiya $(a,b)$ intervalda $\alpha-1/p$ ko`rsatkich bilan Gyolder shartini qanoatlantiradi.

**1.3.–Teorema**[48: 64 bet]**.** Ushbu $0<\alpha<1$, $1<p<1/\alpha$, $q=p/(1-p\alpha)$ shartlar bajarilgan bo`lsin. Agar $\varphi(x)\in L_p(a,b)$ bo`lsa, u holda kasr tartibli $D_{ax}^{-\alpha}\varphi(x)$ va $D_{xb}^{-\alpha}\varphi(x)$ integrallar $L_q(a,b)$ fazoga tegishli bo`lib, quyidagi

$$\left\|D_{ax}^{-\alpha}\varphi(x)\right\|_{L_q(a,b)} \leq C_1\left\|\varphi(x)\right\|_{L_p(a,b)},$$

$$\left\|D_{xb}^{-\alpha}\varphi(x)\right\|_{L_q(a,b)} \leq C_2\left\|\varphi(x)\right\|_{L_p(a,b)}$$

baho o`rinli bo`ladi, bu yerda $C_1$ va $C_2$ – $\alpha$ va $p$ bog'liq o`zgarmaslar. $D_{ax}^{-\alpha}\varphi(x)$ va $D_{xb}^{-\alpha}\varphi(x)$ kasr tartibli integrallar mos ravishda (2.1) va (2.2) formulalar orqali aniqlanadi.

**1. 4.–Teorema**[48: 91 bet]**.** Ushbu $0<\alpha<1$, $1<p<1/\alpha$, $q=p/(1-p\alpha)$ shartlar bajarilgan bo`lsin. Agar $\varphi(x)\in L_p(R^1)$ bo`lsa, u holda kasr tartibli $D_{-\infty x}^{-\alpha}\varphi(x)$ va $D_{x\infty}^{-\alpha}\varphi(x)$ integrallar $L_q(R^1)$ fazoga tegishli bo`lib, quyidagi

$$\left\| D_{-\infty x}^{-\alpha}\varphi(x) \right\|_{L_q(R^1)} \le C_3 \left\| \varphi(x) \right\|_{L_p(R^1)},$$

$$\left\| D_{x\infty}^{-\alpha}\varphi(x) \right\|_{L_q(R^1)} \le C_4 \left\| \varphi(x) \right\|_{L_p(R^1)}$$

baho o`rinli bo`ladi, bu erda $C_3$ va $C_4 -$ $\alpha$ va $p$ bog'liq o`zgarmaslar. $D_{-\infty x}^{-\alpha}\varphi(x)$ va $D_{x\infty}^{-\alpha}\varphi(x)$ kasr tartibli integrallar mos ravishda $a=-\infty$ da (2.61) va $b=\infty$ da (2.62) formulalar orqali aniqlanadi.

**1. 5 -Teorema** [31], [26: 25 bet]**.** Quyidagi $1 \le p < q \le \infty;$ $\eta_0 < 1-\frac{1}{p}$ va $\eta_1 > \alpha - \frac{1}{p}$ tengsizliklar bajarilgan bo`lsin. Agar $x^{\eta_{j-1}} f_{j-1}(x) \in L_p(0,\infty)$ $(j=1,2)$ va $\alpha > \frac{1}{p}-\frac{1}{q}$ $(\alpha = \frac{1}{p}-\frac{1}{q};$ $1 = p < q < \infty$ va $1 < p < q = \infty$ dan tashqari) bo`lsa, u holda

$x^{\frac{1}{p}-\frac{1}{q}-\alpha+\eta_0} I_{0+}^{\alpha} f_0(x) \in L_q(0,\infty)$, $x^{\frac{1}{p}-\frac{1}{q}-\alpha+\eta_1} I_{-}^{\alpha} f_1(x) \in L_q(0,\infty)$ bo`lib, quyidagi

$$\left\| x^{\frac{1}{p}-\frac{1}{q}-\alpha+\eta_0} I_{0+}^{\alpha} f_0(x) \right\|_q \le C_5 \left\| x^{\eta_0} f_0(x) \right\|_p, \tag{2.71}$$

$$\left\| x^{\frac{1}{p}-\frac{1}{q}-\alpha+\eta_1} I_{0-}^{\alpha} f_1(x) \right\|_q \le C_6 \left\| x^{\eta_1} f_1(x) \right\|_p \tag{2.72}$$

baho o`rinli bo`ladi, bu yerda $C_5$ va $C_6 -$ $c$ va $p$ bog'liq o`zgarmaslar. Bundagi $I_{0+}^{\alpha}$ va $I_{0-}^{\alpha}$ - operatorlar Saygo integral operatorlari bo`lib, u kasr tartibli integrallar bilan quyidagicha bog'langan[27: 5-§. Gl. 2]:

$$D_{0x}^{-\alpha} f(x) \equiv I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad 0 < x < \infty, \tag{2.73}$$

$$D_{x\infty}^{-\alpha} f(x) \equiv I_{-}^{\alpha} f(x) = \frac{1}{\Gamma(c)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt, \quad -\infty < x < \infty. \tag{2.74}$$

***Kasr tartibli integral va differensial operatorlarning eng sodda xossalari*** [28: 8 bet]:

$1^0$. Agar $f(x)\in L_1(a,b)$ bo`lsa, u holda ixtiyoriy $\alpha>0$ son va deyarli hamma $x\in(a,b)$ uchun

$$D_{ax}^{\alpha}D_{ax}^{-\alpha}f(x)=f(x),\qquad D_{xb}^{\alpha}D_{xb}^{-\alpha}f(x)=f(x) \qquad (2.75)$$

tengliklar o`rinli bo`ladi.

***Isboti.*** (2.11) va (2.13) formulalarga binoan

$$J=D_{ax}^{\alpha}D_{ax}^{-\alpha}f(x)=\frac{d^n}{dx^n}D_{ax}^{-(n-\alpha)}D_{ax}^{-\alpha}f(x)=$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x\frac{dt}{(x-t)^{\alpha-n+1}}\int_a^t\frac{f(z)}{(t-z)^{1-\alpha}} \qquad (2.76)$$

hamda ushbu

$$\int_a^b dx\int_a^x f(x,y)dy=\int_a^b dy\int_y^b f(x,y)dx \qquad (2.77)$$

Dirixle formulasiga ko`ra (2.16) dan quyidagini hosil qilamiz:

$$J=\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x f(z)dz\int_z^x(x-t)^{n-\alpha-1}(t-z)^{\alpha-1}. \qquad (2.78)$$

Ichki integralni $J_1$ bilan belgilaymiz, ya'ni

$$J_1=\int_z^x(x-t)^{n-\alpha-1}(t-z)^{\alpha-1}dt.$$

Bu integralga $t=z+(x-z)\,\tau$ almashtirish bajarib, quyidagilarni $dt=(x-z)d\tau,\ \ t-z=(x-z)\tau,\ \ x-t=(x-z)(1-\tau)$ e'tiborga olib, $J_1$ ni quyidagicha yozib olamiz:

$$J_1=(x-z)^{n-\alpha-1+\alpha-1+1}\int_0^1\tau^{\alpha-1}(1-\tau)^{n-\alpha-1}d\tau.$$

Bundan va (1.19), (1.23) formulalardan foydalanib. ushbu ifodani olamiz:

$$J_1=(x-z)^{n-1}\frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)},\qquad n>1 \qquad (2.79)$$

(2.79) ni (2.78) ga qo`yib, ushbu

$$J=\left(\frac{d}{dx}\right)^n\frac{1}{\Gamma(n)}\int_a^x(x-z)^{n-\alpha}\ f(z)dz \qquad (2.80)$$

formulani hosil qilamiz. Endi (2.80) ni hisoblaymiz:

$$J=\frac{1}{\Gamma(n)}\left(\frac{d}{dx}\right)^{n-1}\left(\frac{d}{dx}\int_a^x (x-z)^{n-1} f(z)dz\right)=\frac{1}{\Gamma(n)}\left(\frac{d}{dx}\right)^{n-1}\left((n-1)\int_a^x (x-z)^{n-2} f(z)dz\right)=$$

$$=\frac{n-1}{\Gamma(n)}\left(\left(\frac{d}{dx}\right)^{n-2}\int_a^x \frac{d}{dx}(x-z)^{n-2} f(z)dz\right)_{n-2,\ \dots}=$$

$$=\frac{n-1}{\Gamma(n)}\left(\frac{d}{dx}\right)^{n-2}\left((n-2)\int_a^x (x-z)^{n-3} f(z)dz\right)_{n-3,\dots}=\dots=\frac{(n-1)(n-2)(n-3)\dots 2\cdot 1}{\Gamma(n)} f(x).$$

Bundan va (1.11) formulaga ko`ra

$$J=D_{ax}^{\alpha}D_{ax}^{-\alpha} f(x)=\frac{(n-1)\,!}{(n-1)\,!} f(x)=f(x).$$

(2.75) formulaning ikkinchi ayniyati (2.62) va (2.64) lardan foydalanib, xuddi yuqoridagi hisoblashlarga o`xshash isbotlanadi.

**1$^0$ xossa isbotlandi.**

**Izoh:** $\int_a^x (x-t)^{n-1} f(t)dt=(n-1)!\underbrace{\int_a^x dt\int_a^x dt\dots\int_a^x f(t)dt}_{n-\text{марта}}$ tenglik o`rinli.

**2$^0$.** Agar $0<\alpha_1,\alpha_2<\infty$ bo`lsa, u holda $(a,b)$ intervalning deyarli hamma joyida quyidagi tengliklar o`rinli:

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1} f(x)=D_{ax}^{-\alpha_1}D_{ax}^{-\alpha_2} f(x)=D_{ax}^{-(\alpha_1+\alpha_2)} f(x). \quad (2.81)$$

***I s b o t i.*** (2.1) formulaga ko`ra quyidagi

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1} f(x)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x (x-t)^{\alpha_2-1}dt\int_a^t (t-z)^{\alpha_1-1} f(z)dz \quad (2.82)$$

tenglikni olamiz.

(2.77) formulaga ko`ra esa (2.82) ni ushbu

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1} f(x)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x f(z)dz\int_z^x (x-t)^{\alpha_2-1}(t-z)^{\alpha_1-1}dt \quad (2.83)$$

ko`rinishda ifodalaymiz.

(2.83) teglikni o`ng tomonidagi ichki integralda

$$t=z+(x-z)\tau,\quad dt=(x-z)d\tau,\quad (x-t)=(x-t)(1-\tau),\quad t-z=(x-z)\tau \quad (2.84)$$

almashtirish bajarib, so`ng (1.19), (1.23) formulalardan foydalanib, (2.83) ni quyidagicha yozib olamiz:

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x(x-z)^{\alpha_1+\alpha_2-1}f(z)dz\int_0^1\tau^{\alpha_1-1}(1-\tau)^{\alpha_2-1}d\tau=$$

$$=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2)}\int_a^x(x-z)^{\alpha_1+\alpha_2-1}f(z)dz=$$

$$=\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^x(x-z)^{\alpha_1+\alpha_2-1}f(z)dz.$$

Bundan va (1.164) formulaga ko`ra ushbu

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x)=D_{ax}^{-(\alpha_1+\alpha_2)}f(x) \quad (2.85)$$

tenglikni olamiz.

Xuddi shunday

$$D_{ax}^{-\alpha_1}D_{ax}^{-\alpha_2}f(x)=D_{ax}^{-(\alpha_1+\alpha_2)}f(x) \quad (2.86)$$

tenglikni isbotlash mumkin.

(2.25) va (2.26) dan $2^0$ xossaning isboti kelib chiqadi.

**$3^0$.** (1.164) va (1.165) dan ushbu tenglik kelib chiqadi:

$$D_{ax}^{0}f(x)=f(x),\quad D_{bx}^{0}f(x)=f(x). \quad (2.87)$$

**$4^0$.** Agar $f(x)\in L_1(a,b)$ bo`lsa, u holda

1. agar $\beta\geq\alpha>0$ bo`lsa, u holda

$$D_{ax}^{\alpha}D_{ax}^{-\beta}f(x)=D_{ax}^{-(\beta-\alpha)}f(x),\quad x\in(a,b) \quad (2.88)$$

2. agar $\alpha>\beta\geq 0$ va $f(x)$ funksiya $(a,b)$da kasr tartibli $D_{ax}^{\alpha-\beta}f(x)$ hosilaga ega bo`lsa, u holda

$$D_{ax}^{\alpha}D_{ax}^{-\beta}f(x)=D_{ax}^{\alpha-\beta}f(x),\quad x\in(a,b) \quad (2.89)$$

***1-isboti.*** (2.61) va (2.62) ta’rifga ko`ra hamda (2.87) formulani e’tiborga olib, (2.89) tenglikni o`ng tomonidan quyidagini olamiz:

$$D_{ax}^{\alpha}D_{ax}^{-\beta}f(x)=\frac{d}{dx}D_{ax}^{\alpha-1}D_{ax}^{-\beta}f(x)=\frac{1}{\Gamma(\beta)\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}dt\times$$

$$\times\int_a^x(t-z)^{\beta-1}f(z)dz=\frac{1}{\Gamma(\beta)\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x f(z)dz\int_z^x(x-t)^{-\alpha}(t-z)^{\beta-1}dt. \quad (2.90)$$

(2.83) almashtirish va (1.19), (1.23) formulalarga ko`ra (2.90) formuladan quyidagini hosil qilamiz:

$$D_{ax}^{\alpha}D_{ax}^{-\beta}f(x)=\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1-\alpha+\beta)}\frac{d}{dx}\int_a^x(x-t)^{\beta-\alpha}f(z)dz \quad (2.91)$$

$\beta-\alpha>0$ shartiga ko`ra (2.91) formuladagi integralni differensiallash mumkin, ya'ni

$$D_{ax}^{\alpha}D_{ax}^{-\beta}f(x)=\frac{1}{\Gamma(1-\alpha+\beta)}\left[(x-z)^{\beta-\alpha}f(z)\Big|_{z=x}+(\beta-\alpha)\int_a^x(x-z)^{\beta-\alpha-1}f(z)dz\right]=$$

$$=\frac{\beta-\alpha}{(\beta-\alpha)\Gamma(\beta-\alpha)}\int_a^x(x-t)^{\beta-\alpha-1}f(z)dz=D_{ax}^{(\beta-\alpha)}f(x).$$

**$4^0$ xossaning 1- holi isbotlandi.** 2-holi ham xuddi shunday isbotlanadi.

**$5^0$.** Agar $D_{ax}^{\alpha}f(x)\in L_1(a,b)$ mavjud bo`lsa, u holda $(a,b)$ ni deyarli hamma joyida quyidagi ayniyat o`rinli:

$$D_{ax}^{-\alpha}D_{ax}^{\alpha}f(x)=f(x)-\sum_{k=1}^{n}\left[D_{ax}^{\alpha-\kappa}f(x)\right]_{x=a}\frac{(x-a)^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)},\quad n-1<\alpha\le n. \quad (2.92)$$

Agar $n=1$ bo`lib, $0<\alpha<1$ bo`lsa, u holda ushbu

$$D_{ax}^{-\alpha}D_{ax}^{\alpha}f(x)=f(x)-\left[D_{ax}^{\alpha-1}f(x)\right]_{x=a}\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \quad (2.93)$$

tenglik o`rinlidir.

**$6^0$. Kasr tartibli differensial operator uchun ekstremum prinsipi.** Agar ushbu

a) $\omega(t)$ kamaymaydigan, musbat va $[a,b]$ kesmada uzluksiz funksiya;

b) $f(t)$ funksiya $[a,b]$ kesmada uzluksiz funksiya;

shartlar bajarilsa, u holda quyidagi lemma o`rinli:

**1.1-Lemma.** Agar $f(t)$ funksiya $\left[a,b\right]$ kesmada o`zining musbat maksimumiga (manfiy minimumiga) $t=x,\ a<x<b$ nuqtada erishsa va $t=x$ nuqtaning ixtiyoriy kichik atrofida $\omega(t)f(t)$ ko`paytma $\gamma>\alpha$ ko`rsatkich bilan Gyolder shartini qanoatlantirsa, u holda

$$D_{ax}^{\alpha}\omega(x)f(x)>0, \qquad \left(D_{ax}^{\alpha}\omega(x)f(x)<0\right) \tag{2.94}$$

tengsizliklar o`rinli.

***1.1-lemmaning isboti.*** $D_{ax}^{\alpha}\omega(x)f(x)$ operatorni (2.63) formulaga ko`ra quyidagicha yozib olamiz:

$$\Gamma(1-\alpha)D_{ax}^{\alpha}\omega(x)f(x)=\Gamma(1-\alpha)\frac{d}{dx}\left[D_{ax}^{\alpha-1}\omega(x)f(x)\right]=$$

$$=\frac{d}{dx}\int_a^x\frac{\omega(t)f(t)dt}{(x-t)^{\alpha}}=\frac{d}{dx}\int_a^x\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{\alpha}}dt+\frac{d}{dx}\int_a^x\frac{\omega(x)f(x)dt}{(x-t)^{\alpha}}.$$

Ushbu funksiyani

$$F_{\varepsilon}(x)=\frac{d}{dx}\int_a^{x-\varepsilon}\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{\alpha}}dt+\frac{d}{dx}\int_a^{x-\varepsilon}\frac{\omega(x)f(x)dt}{(x-t)^{\alpha}} \tag{2.95}$$

qaraymiz. Bunda

$$F(x)=\lim_{\varepsilon\to 0}F_{\varepsilon}(x)=\Gamma(1-\alpha)D_{ax}^{\alpha}\omega(x)f(x). \tag{2.96}$$

(2.95) ifodada qatnashgan integrallarni differensiallab, quyidagi tenglikni olamiz:

$$F_{\varepsilon}(x)=\frac{\omega(x-\varepsilon)f(x-\varepsilon)-\omega(x)f(x)}{\varepsilon^{\alpha}}-\int_a^{x-\varepsilon}\frac{\left[\omega(x)f(x)\right]'_x dt}{(x-t)^{\alpha}}-$$

$$-\alpha\int_a^{x-\varepsilon}\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{\alpha+1}}dt+\int_a^{x-\varepsilon}\frac{\left[\omega(x)f(x)\right]'_x dt}{(x-t)^{\alpha}}-$$

$$-\alpha\int_a^{x-\varepsilon}\frac{\omega(x)f(x)dt}{(x-t)^{\alpha+1}}+\frac{\omega(x)f(x)}{\varepsilon^{\alpha}}.$$

Bundan $\varepsilon\to 0$ limitga o`tib va ushbu

$$\int_a^{x-\varepsilon}\frac{\omega(x)f(x)dt}{(x-t)^{\alpha+1}}=\frac{\omega(x)f(x)}{\alpha\,\varepsilon^{\alpha}}-\frac{\omega(x)f(x)}{\alpha(x-a)^{\alpha}},$$

$$\lim_{\varepsilon\to 0}\frac{\omega(x-\varepsilon)\,f(x-\varepsilon)-\omega(x)\,f(x)}{\varepsilon^{\alpha}}=0$$

ayniyatlarni hamda (2.96) belgilashni e'tiborga olib, quyidagi

$$F(x)=\Gamma(1-\alpha)D_{ax}^{\alpha}\omega(x)\,f(x)=\frac{\omega(x)\,f(x)}{(x-a)^{\alpha}}+$$

$$+\alpha\int_a^{\theta}\frac{\omega(x)\,f(x)-\omega(t)\,f(t)}{(x-t)^{\alpha+1}}dt+\alpha\int_{\theta}^{x}\frac{\omega(x)\,f(x)-\omega(t)\,f(t)}{(x-t)^{\alpha+1}}dt \quad (2.37)$$

tenglikka ega bo`lamiz.

(2.97) munosabatdan va 1.1 lemmaning shartlaridan (2.84) tengsizlikni to`g'riligi kelib chiqadi.

***1.1 lemma isbot bo`ldi.***

## 2.3-§. Kasr tartibli integral va differensial operatorlar kompozitsiyasi natijasida hosil bo`lgan ayniyatlar

**1.2- Lemma.** Agar $0<\alpha,\beta<1$ va $(x-a)^{-\alpha}f(x),\ (x-a)^{-\beta}f(x)\in L_1(a,b)$ bo`lsa, u holda $(a,b)$ intervalni deyarli hamma joyida quyidagi ayniyat o`rinlidir:

$$D_{ax}^{-\beta}(x-a)^{-\beta}D_{ax}^{-\alpha}(x-a)^{-\alpha}f(x)=D_{ax}^{-\alpha}(x-a)^{-\alpha}D_{ax}^{-\beta}(x-a)^{-\beta}f(x). \quad (2.98)$$

***1.2- lemmaning isboti.*** (2.61) formulaga ko`ra

$$\mathrm{E}_1(x)=D_{ax}^{-\beta}(x-a)^{-\beta}D_{ax}^{-\alpha}(x-a)^{-\alpha}f(x)=$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}(t-a)^{-\beta}dt\int_a^t(t-\xi)^{\alpha-1}(\xi-a)^{-\alpha}f(\xi)d\xi=$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x(\xi-a)^{-\alpha}d\xi\int_{\xi}^{x}(x-t)^{\beta-1}(t-\xi)^{\alpha-1}(t-a)^{-\beta}dt. \quad (2.99)$$

(2.99) formulaning ichki integralida quyidagicha almashtirish qilamiz:

$$t=\xi+(x-\xi)z,\ t-a=\xi-a+(x-\xi)z=(\xi-a)\left(1-\frac{\xi-x}{\xi-a}\right),\quad t-\xi=(x-\xi)z,$$

$$x-t=(\xi-x)(1-z),\qquad dt=(\xi-x)dz,\quad t=\xi\Rightarrow z=0,\quad t=x\Rightarrow z=1.\quad (2.100)$$

U holda (2.99) quyidagi ko`rinishni oladi:

$$\mathrm{E}_1(x)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\frac{\xi^{-\alpha-\beta}}{(x-\xi)^{1-\alpha-\beta}}f(\xi)d\xi\int_0^1\frac{z^{\alpha-1}}{(1-z)^{1-\beta}}\left[1+\frac{x-\xi}{\xi}z\right]^{-\beta}dz.\quad (2.101)$$

(1.63) formulaga ko`ra (2.99) ni quyidagicha yozamiz:

$$\mathrm{E}_1(x)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_a^x\frac{\xi^{-\alpha-\beta}f(\xi)}{(x-\xi)^{1-\alpha-\beta}}F\left(\alpha,\beta,\alpha+\beta;1-\frac{x}{\xi}\right)d\xi.\ (2.102)$$

Xuddi shunday (1.58) formulaga ko`ra $\alpha$ va $\beta$ ni o`rnini almashtirib, (2.102) formuladan quyidagi

$$\mathrm{E}_1(x)=D_{ax}^{-\alpha}x^{-\alpha}D_{ax}^{-\beta}x^{-\beta}f(x)=D_{ax}^{-\beta}x^{-\beta}D_{ax}^{-\alpha}x^{-\alpha}f(x)=$$

$$=\frac{1}{\Gamma(\alpha+\beta)}\int_a^x x^{-\alpha-\beta}(x-\xi)^{\alpha+\beta-1}F\left(\beta,\alpha,\alpha+\beta,1-\frac{x}{\xi}\right)d\xi$$

tenglikni hosil qilamiz. Bu esa (2.98) ni to`g’riligini isbotlaydi.

***1.2 lemma isbotlandi.***

**1.3–Lemma.** Agar $0<2\alpha<1$ va $(x-a)^{-\alpha}f(x),\ (b-x)^{-\alpha}f(x)\in L_1(a,b)$ bo`lsa, u holda $(a,b)$ intervalning deyarli hamma joyida quyidagi ayniyatlar o`rinlidir:

$$D_{ax}^{\alpha}(x-a)^{2\alpha-1}D_{ax}^{\alpha-1}(x-a)^{-\alpha}f(x)=(x-a)^{\alpha-1}D_{ax}^{2\alpha-1}f(x),\qquad (2.103)$$

$$D_{xb}^{\alpha}(b-x)^{2\alpha-1}D_{xb}^{\alpha-1}(b-x)^{-\alpha}f(x)=(b-x)^{\alpha-1}D_{xb}^{2\alpha-1}f(x)\ .\qquad (2.104)$$

***1.3 lemmaning isboti.*** (2.103) ayniyatni chap tomonini $\mathrm{E}_2(x)$ bilan belgilab, (2.61), (2.63) va (2.97) formulalarga ko`ra quyidagi tenglikni hosil qilamiz:

$$\mathrm{E}_2(x)=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}(t-a)^{2\alpha-1}dt\int_a^t(t-\xi)^{-\alpha}(\xi-a)^{-\alpha}f(\xi)d\xi=$$

$$=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x(\xi-a)^{-\alpha}f(\xi)d\xi\int_\xi^x(t-a)^{2\alpha-1}(x-t)^{-\alpha}(t-\xi)^{-\alpha}dt. \quad (2.105)$$

(2.105) tenglikning o`ng tomonidagi ichki integralga (2.44) almashtirishni bajarib, uni ushbu

$$\mathrm{E}_2(x)=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x\frac{(x-\xi)^{1-2\alpha}f(\xi)}{(\xi-a)^{1-\alpha}}d\xi\int_0^1 z^{-\alpha}(1-z)^{-\alpha}\left(1-\frac{\xi-x}{\xi-a}z\right)^{2\alpha-1}dz \quad (2.106)$$

ko`rinishda yozib olamiz. (1.63) formulaga ko`ra (2.106) tenglikdan quyidagini hosil qilamiz:

$$\mathrm{E}_2(x)=\frac{1}{\Gamma^2(1-\alpha)}\cdot\frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_a^x\frac{(x-\xi)^{1-2\alpha}}{(\xi-a)^{1-\alpha}}F\left(1-\alpha,1-2\alpha,2-2\alpha,\frac{\xi-x}{\xi-a}\right)f(\xi)d\xi=$$

$$=\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_a^x\frac{(x-\xi)^{1-2\alpha}}{(\xi-a)^{1-\alpha}}F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{\xi-x}{\xi-a}\right)f(\xi)d\xi. \quad (2.107)$$

Endi (1.51), (1.59) va (1.98) formulalarga ko`ra (2.107) dan quyidagi tenglikka kelamiz:

$$\mathrm{E}_2(x)=\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_a^x\frac{(x-\xi)^{1-2\alpha}}{(\xi-a)^{1-\alpha}}\left(\frac{x-a}{\xi-a}\right)^{2\alpha-1}F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{x-\xi}{x-a}\right)f(\xi)d\xi=$$

$$=\frac{1-2\alpha}{\Gamma(2-2\alpha)}\int_a^x(\xi-a)^{-\alpha}\left(\frac{x-\xi}{x-a}\right)^{-2\alpha}\left(\frac{x-\xi}{x-a}\right)f(\xi)F\left(1-\alpha,2-2\alpha,2-2\alpha;\frac{x-\xi}{\xi-a}\right)d\xi=$$

$$=\frac{1-2\alpha}{\Gamma(2-2\alpha)}\int_a^x\left(\frac{x-\xi}{x-a}\right)^{-2\alpha}\frac{x-a-x+\xi}{(\xi-a)^{\alpha}(x-a)^2}f(\xi)F\left(1-\alpha,2-2\alpha,2-2\alpha;\frac{x-\xi}{x}\right)d\xi=$$

$$=\frac{1-2\alpha}{\Gamma(2-2\alpha)}\int_a^x(\xi-a)^{1-\alpha}(x-\xi)^{-2\alpha}(x-a)^{-2+2\alpha}\left(\frac{\xi-a}{x-a}\right)^{\alpha-1}f(\xi)d\xi. \quad (2.108)$$

(1.6) va (2.61) formulalarga ko`ra (2.108) ni quyidagicha

$$\mathrm{E}_2(x)=\frac{(x-a)^{\alpha-1}}{\Gamma(1-2\alpha)}\int_a^x(x-\xi)^{-2\alpha}f(\xi)d\xi=(x-a)^{\alpha-1}D_{ax}^{2\alpha-1}f(x)$$

yozib olamiz.

Bundan esa (2.103) ning to`g'riligi kelib chiqadi.

Endi (2.104) ayniyatni isbotlaymiz. (2.104) ayniyatni chap tomonini $E_3(x)$ deb belgilab, (2.62), (2.64) va (2.97) formulaga ko`ra $n=1,\ 0<\alpha<0,5$ ni e'tiborga olib, quyidagiga

$$E_3(x)=-\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_x^b (t-x)^{-\alpha}(b-t)^{2\alpha-1}dt\int_t^b (\xi-t)^{-\alpha}(b-\xi)^{-\alpha}d\xi=$$

$$=-\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_x^b (b-\xi)^{-\alpha} f(\xi)d\xi\int_x^\xi (t-x)^{-\alpha}(\xi-t)^{-\alpha}(b-t)^{2\alpha-1}dt \quad (2.109)$$

ega bo`lamiz.

(2.109) tenglikni ichki integralida $t=\xi-(\xi-x)z$ almashtirishni bajarib va ushbu

$$\xi-t=(\xi-x)z,\quad t-x=(\xi-x)(1-z),\quad dt=-(\xi-x)dz,$$

$$b-t=(b-\xi)+(\xi-x)z,\quad t=\xi\ \Rightarrow\ z=0,\quad t=x\ \Rightarrow\ z=1 \qquad (2.110)$$

tengliklarni e'tiborga olib, quyidagi

$$E_3(x)=-\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_x^b \frac{(\xi-x)^{1-2\alpha}}{(b-\xi)^{1-\alpha}} f(\xi)d\xi\int_0^1 z^{-\alpha}(1-z)^{-\alpha}\left(1+\frac{\xi-x}{b-\xi}z\right)^{2\alpha-1}dz \qquad (2.111)$$

tenglikni hosil qilamiz.

(1.63) formulaga ko`ra esa, (2.111) formulani ushbu ko`rinishda yozib olamiz:

$$E_3(x)=-\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_x^b \frac{(\xi-x)^{1-2\alpha} f(\xi)}{(b-\xi)^{1-\alpha}} F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{x-\xi}{b-\xi}\right)d\xi. \quad (2.112)$$

Endi (2.112) formulaga (1.111) formulani qo`llab, uni quyidagi ko`rinishda ifodalaymiz:

$$E_3(x)=-\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_x^b (b-\xi)^{\alpha-1}(\xi-x)^{1-2\alpha}\left(\frac{b-x}{b-\xi}\right)^{2\alpha-1}\times$$

$$\times F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{\xi-x}{b-x}\right) f(\xi)d\xi=-\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}\int_x^b (b-\xi)^{-\alpha}\left(\frac{\xi-x}{b-x}\right)^{1-2\alpha}\times$$

$$\times F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{\xi-x}{b-x}\right)f(\xi)d\xi=-\frac{1}{\Gamma(2-2\alpha)}\int_x^b(b-\xi)^{-\alpha}f(\xi)\times$$

$$\times\frac{d}{dx}\left[\left(\frac{\xi-x}{b-x}\right)^{1-2\alpha}F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{\xi-x}{b-x}\right)\right]d\xi. \qquad (2.113)$$

(2.113) formulaga (1.98) formulani qo`llab, (1.6) va (1.59) ni e'tiborga olib, (2.113) formulani ushbu ko`rinishda yozib olamiz:

$$\mathrm{E}_3(x)=-\frac{1-2\alpha}{\Gamma(2-2\alpha)}\int_x^b f(\xi)(b-\xi)^{-\alpha}\left(\frac{\xi-x}{b-x}\right)^{-2\alpha}\left(1-\frac{\xi-x}{b-x}\right)^{\alpha-1}\frac{x-b+\xi-x}{(b-x)^2}d\xi=$$

$$=\frac{1}{\Gamma(1-2\alpha)}\int_x^b(b-\xi)^{-\alpha}\left(\frac{\xi-x}{b-x}\right)^{-2\alpha}\left(\frac{b-\xi}{b-x}\right)^{\alpha-1}\frac{(b-\xi)}{(b-x)^2}f(\xi)d\xi=$$

$$=\frac{(b-x)^{\alpha-1}}{\Gamma(1-2\alpha)}\int_x^b(\xi-x)^{-2\alpha}f(\xi)d\xi. \qquad (2.114)$$

(2.62) va (2.114) dan

$$\mathrm{E}_3(x)=(b-x)^{\alpha-1}D_{xb}^{2\alpha-1}f(x)$$

(2.114) ayniyat kelib chiqadi.

***1.3 lemma isbotlandi.***

**1.4-Lemma.** Agar $0<2\beta<1$ va $(x-a)^{\beta-1}f(x)$, $(b-x)^{\beta-1}f(x)\in L_1(a,b)$ bo`lsa, u holda $(a,b)$ intervalning deyarli hamma joyida quyidagi ayniyatlar o`rinlidir:

$$D_{ax}^{1-\beta}(x-a)^{1-2\beta}D_{0x}^{-\beta}(x-a)^{\beta-1}f(x)=(x-a)^{-\beta}D_{0x}^{1-2\beta}f(x), \qquad (2.115)$$

$$D_{xb}^{1-\beta}(b-x)^{1-2\beta}D_{xb}^{-\beta}(b-x)^{\beta-1}f(x)=(b-x)^{-\beta}D_{xb}^{1-2\beta}f(x). \qquad (2.116)$$

***1.4 lemmaning isboti.*** (2.113) ayniyatni chap tomonini $\mathrm{E}_4(x)$ bilan belgilab, (2.61) va (2.63) formulalardan foydalanib, **1.3-lemmaga** o`xshash $\mathrm{E}_4(x)$ uchun ushbu

$$\mathrm{E}_4(x)=\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int_a^x(\xi-a)^{\beta-1}\left(\frac{x-\xi}{x-a}\right)^{2\beta-1}F\left(2\beta-1,\beta,2\beta;\frac{x-\xi}{x-a}\right)f(\xi)d\xi \quad (2.117)$$

ifodani hosil qilamiz.

Quyidagi funksiyani qaraymiz:

$$\mathrm{E}_{4\varepsilon}(x)=\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int_0^{x-\varepsilon}(\xi-a)^{\beta-1}\left(\frac{x-\xi}{x-a}\right)^{2\beta-1}F\left(2\beta-1,\beta,2\beta;\frac{x-\xi}{x-a}\right)f(\xi)d\xi \quad (2.118)$$

Ushbu

$$\mathrm{E}_4(x)=\lim_{\varepsilon\to 0}\mathrm{E}_{4\varepsilon}(x)=\frac{(x-a)^{-\beta}}{\Gamma(2\beta)}\frac{d}{dx}\int_a^x(x-\xi)^{2\beta-1}f(\xi)d\xi \qquad (2.119)$$

ifodani to`g'riligini isbotlaymiz.

(2.119) tenglikni o`ng tomonini differensiallab va (1.59), (1.98) formulalarni qo`llab, ushbu

$$\mathrm{E}_{4\varepsilon}(x)=\frac{(x-\varepsilon-a)^{\beta-1}}{\Gamma(2\beta)}\left(\frac{\varepsilon}{x-a}\right)^{2\beta-1}F\left(2\beta-1,\beta,2\beta;\frac{\varepsilon}{x-a}\right)f(x-\xi)+$$

$$+\frac{2\beta-1}{\Gamma(2\beta)}(x-a)^{-\beta}\int_a^{x-\varepsilon}(x-\xi)^{2\beta-2}f(\xi)d\xi \qquad (2.120)$$

tenglikni hosil qilamiz.

Quyidagi

$$(2\beta-1)\int_a^{x-\varepsilon}(x-\xi)^{2\beta-2}f(\xi)d\xi=\frac{d}{dx}\int_a^{x-\varepsilon}f(\xi)(x-\xi)^{2\beta-1}d\xi-\varepsilon^{2\beta-1}f(x-\xi) \qquad (2.121)$$

tenglik o`rinlidir.

(2.120) ifodaga (2.121) ayniyatni qo`llab, ushbu

$$\mathrm{E}_{4\varepsilon}(x)=\frac{\varepsilon^{2\beta-1}(x-a)^{\beta}}{\Gamma(2\beta)}\left[\left(\frac{x-a}{x-\xi}\right)F\left(2\beta-1,\beta,2\beta;\frac{\varepsilon}{x-a}\right)-1\right]f(x-\xi)+$$

$$+\frac{(x-a)^{-\beta}}{\Gamma(2\beta)}\frac{d}{dx}\int_a^{x-\varepsilon}(x-\xi)^{2\beta-1}f(\xi)d\xi \qquad (2.122)$$

ifodani hosil qilamiz.

Agar (2.112) dan $\varepsilon\to 0$ intiltirib limitga o`tib, so`ng

$F\left(2\beta-1,\beta,2\beta;\frac{\varepsilon}{x-a}\right)=1+O\left(\frac{\varepsilon}{x-a}\right)$ tenglikni hisobga olgan holda, quyidagi

$$\mathrm{E}_4(x)=\frac{(x-a)^{-\beta}}{\Gamma(2\beta)}\frac{d}{dx}\int_a^x(x-\xi)^{2\beta-1}f(\xi)d\xi$$

formulani hosil qilamiz.

Bundan va (2.123) formulaga ko`ra quyidagini olamiz:

$$\mathrm{E}_4(x)=(x-a)^{-\beta}D_{ax}^{1-2\beta}f(x).$$

Bu (2.115) ayniyatni to`g'riligini isbotlaydi. (2.114) ayniyat ham xuddi yuqoridagiga o`xshash isbotlanadi.

***1.4 lemma isbotlandi.***

$a$, $b$, $c$ **haqiqiy sonlar ma'lum bir shartlarni qanoatlantirganda kasr tartibli integral operatorlar kompozitsiyasi natijasida hosil bo`lgan ushbu ayniyatlar**

**o`rinlidir:**

$$F_{0x}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=x^a I_{0+}^b x^{-a} I_{0+}^{c-b} f(x)=x^b I_{0+}^{c-b} x^{a-c} I_{0+}^b x^{c-a-b} f(x), \quad (2.123)$$

$$F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=x^a I_{-}^b x^{-a} I_{-}^{c-b} f(x)=x^{-a} I_{-}^{c-b} x^{a-c} I_{-}^b x^{c-a-b} f(x), \quad (2.124)$$

yoki

$$F_{0x}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=x^a I_{0+}^{c-a} x^{b-c} I_{0+}^a x^{c-a-b} f(x)=x^b I_{0+}^a x^{-b} I_{0+}^{c-a} f(x),$$

$$F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=x^a I_{-}^{c-a} x^{b-c} I_{-}^{a-b} x^{c-a-b} f(x)=x^b I_{-}^a x^{-b} I_{-}^{c-a} f(x),$$

bu yerda

$$F_{0x}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=\frac{1}{\Gamma(c)}\int_0^x f(t)(x-t)^{c-1} F\left(a, b, c; \ \frac{x-t}{x}\right)dt, \ \ c>0, \quad (2.125)$$

$$F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)=\frac{1}{\Gamma(c)}\int_x^\infty f(t)(t-x)^{c-1} F\left(a, b, c; \ \frac{x-t}{x}\right)dt, \ \ c>0, \quad (2.126)$$

$I_{0+}^c[\cdot]$ va $I_{-}^c[\cdot]$ operatorlar mos ravishda (2.73) va (2.74) orqali aniqlanadi. $F_{0x}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)$ va $F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x)$ operatorlar $c>0$ bo`lganda **umumlashgan kasr tartibli** integral operatorlar deyiladi[26: 25-26 betlar], [31].

**(2.123) va (2.124) ayniyatlarning o`rinligi quyidagi lemmalar orqali isbotlanadi:**

**1.5-Lemma[31].** Ushbu

(I) $c \ge b + \frac{1}{p} - \frac{1}{r} \ge \frac{1}{p} - \frac{1}{q}; \quad p \le r \le q$ yoki

(II) $c \ge c - a + \frac{1}{p} - \frac{1}{r} \ge \frac{1}{p} - \frac{1}{q}; \quad p \le r \le q$ (2.127)

(agar $1 = p < q < \infty$ yoki $1 < p < q = \infty$, u holda (I) va (II) tengsizliklarda tenglik belgisi olinmaydi) $1 \le p < q \le \infty$ va $\eta < min\{0,\ c - a - b\} - \frac{1}{p} + 1$ shartlar bajarilsin. Agar $x^{\eta} f(x) \in L_p(0,\infty)$ bo`lsa, u holda $x^{\frac{1}{p}-\frac{1}{q}-c+\eta} F_{0x}\begin{bmatrix} a, b \\ c;\ x \end{bmatrix} f(x) \in L_q(0,\infty)$ bo`lib, quyidagi

$$\left\| x^{\frac{1}{p}-\frac{1}{q}-c+\eta} F_{0x}\begin{bmatrix} a, b \\ c;\ x \end{bmatrix} f(x) \right\|_q \le C_3 \left\| x^{\eta} f(x) \right\|_p \qquad (2.128)$$

baho o`rinli bo`ladi, bu yerda $C_3$ o`zgarmas son $a, b, c$ va $p$ parametrlarga bog'liq.

***1.5 lemmaning isboti.*** Agar $1 = p < q < \infty$, $c \ge b + \frac{1}{p} - \frac{1}{r} \ge \frac{1}{p} - \frac{1}{q}$ va $p < r < q$ bo`lsa, u holda (2.128) dan quyidagi

$$x^{-a} F_{0x}\begin{bmatrix} a, b \\ c;\ x \end{bmatrix} f(x) = I_{0+}^{b} x^{-a} I_{0+}^{c-b} f(x) \qquad (2.129)$$

tenglikni olamiz.

Endi $p < r < q$, $\eta < 1 - \frac{1}{p}$, $c - b \ge \frac{1}{p} - \frac{1}{r}$ bo`lganda $x^{\eta} f(x) \in L_p$ ni e'tiborga olib, (2.69) ayniyatga 1.3 teoremani qo`llab, ushbu tasdiqni olamiz: $x^{\frac{1}{p}-\frac{1}{r}+b-c+\eta} I_{0+}^{c-b} f(x) \in L_r$ tegishli bo`lib, uning uchun quyidagi

$$\left\| x^{\frac{1}{p}-\frac{1}{r}+b-c+\eta} I_{0+}^{c-b} f(x) \right\|_r \le C_3 \left\| x^{\eta} f(x) \right\|_p \qquad (2.130)$$

baho o`rinli bo`ladi.

Ikkinchi tomondan esa $\eta < c-a-b-\frac{1}{p}+1$ yoki $\frac{1}{p}-\frac{1}{r}+a+b-c+\eta < 1-\frac{1}{r}$ shartlar bajarilsa, u holda $x^{\frac{1}{p}-\frac{1}{r}+a+b-c+\eta}\,\overline{f}(x) \in L_r$ bo`ladi, bu yerda

$$\overline{f}(x) = x^{-a} I_{0+}^{c-b} f(x)\,. \tag{2.131}$$

Bundan va (2.129) tenglikdan va 1.3 teoremadan foydalanib, quyidagi xulosaga kelamiz:

$x^{\frac{1}{p}-\frac{1}{q}+a-c+\eta} I_{0+}^{b}\, \overline{f}(x) \in L_q$ tegishli bo`lib, $b \geq \frac{1}{r}-\frac{1}{q}$ bo`lganda ushbu

$$\left\| x^{\frac{1}{p}-\frac{1}{r}+a-c+\eta} I_{0+}^{b}\, \overline{f}(x) \right\|_q \leq C_3 \left\| x^{\frac{1}{p}-\frac{1}{r}+a+b-c+\eta}\, \overline{f}(x) \right\|_r \tag{2.132}$$

baho o`rinli bo`ladi.

(2.130) tengsizlik va (2.131) tenglikni (2.132) tengsizlikka qo`llab (2.128) tengsizlikni hosil qilamiz, ya’ni

$$\left\| x^{\frac{1}{p}-\frac{1}{q}-c+\eta} F_{0x}\begin{bmatrix} a,\, b \\ c;\;\; x \end{bmatrix} f(x) \right\|_r \leq C_3 \left\| x^{\eta} f(x) \right\|_p, \quad \eta < min\{0, c-a-b\} - \frac{1}{p}+1.$$

$p$ va $q$ parametrlarning boshqa qiymatlarida 1.5 lemmaning isboti yuqorida keltirilgan usul yordamida isbotlanadi.

(2.123) va (2.124) ayniyatlarning boshqa hollarida 1.5 lemmaning isboti

$$F_{0x}\begin{bmatrix} a,\, b \\ c;\;\; x \end{bmatrix} f(x) = F_{0x}\begin{bmatrix} b,\, a \\ c;\;\; x \end{bmatrix} f(x),$$

$$F_{0x}\begin{bmatrix} a,\, b \\ c;\;\; x \end{bmatrix} x^{a+b-c} f(x) = x^{a+b-c} F_{0x}\begin{bmatrix} c-a,\, c-b \\ c;\;\;\; x \end{bmatrix} f(x)$$

ayniyatlarga asoslanib isbotlanadi.

Endi (1.226) ayniyatning to`g’riligini isbotlaymiz. Buning uchun ushbu

$$E_5(x) = x^{a} I_{0+}^{b} x^{-a} I_{0+}^{c-b} f(x) \tag{2.133}$$

belgilashni kiritamiz.

(2.73) formulani (2.133) tenglikka qo`llab, quyidagiga ega bo`lamiz:

$$E_5(x)=\frac{x^a}{\Gamma(b)\Gamma(c-b)}\int_0^x (x-t)^{b-1}t^{-a}dt\int_0^t f(z)(t-z)^{c-b-1}dz=$$

$$=\frac{x^a}{\Gamma(b)\Gamma(c-b)}\int_0^x f(z)dz\int_z^x t^{-a}(x-t)^{b-1}(t-z)^{c-b-1}dt. \qquad (2.134)$$

(2.134) tenglikni ichki integraliga $t=z+(x-z)\sigma$ almashtirishni bajarib, quyidagi $t-z=(x-z)\sigma,\ x-t=(x-z)(1-\sigma),$

$$dt=(x-z)d\sigma, \qquad t=z\left(1+\frac{x-z}{z}\sigma\right), \qquad t=z\ \Rightarrow\ \sigma=0, \quad t=x\ \Rightarrow\ \sigma=1$$

tengliklarni e’tiborga olib, ushbu

$$E_5(x)=\frac{x^a}{\Gamma(b)\Gamma(c-b)}\int_0^x (x-z)^{c-1}z^{-a}f(z)dz\int_0^1 \sigma^{c-b-1}(1-\sigma)^{b-1}\left(1+\frac{x-z}{z}\sigma\right)^{-a}d\sigma=$$

$$=\frac{1}{\Gamma(b)\Gamma(c-b)}\int_0^x (x-z)^{c-1}f(z)dz\int_0^1 \sigma^{b-1}(1-\sigma)^{c-b-1}\left(1-\frac{x-z}{x}\sigma\right)^{-a}d\sigma$$

tenglikni hosil qilamiz.

Bundan va (2.123), (2.125) formulalarga asoslanib, ushbu

$$E_5(x)=\frac{1}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\int_0^x f(z)(x-z)^{c-1}F\left(a,b,c,\frac{x-z}{z}\right)dz=$$

$$=\frac{1}{\Gamma(c)}\int_0^x f(z)(x-z)^{c-1}F\left(a,b,c,\frac{x-z}{z}\right)dz=F_{0x}\begin{bmatrix}a,\ b\\ c;\ \ x\end{bmatrix}f(x)$$

tenglikni olamiz.

Xuddi yuqoridagiga o`xshash quyidagi

$$E_6(x)\equiv x^b I_{0+}^{c-b}x^{a-c}I_{0+}^{b}x^{c-a-b}f(x)=F_{0x}\begin{bmatrix}a,\ b\\ c;\ \ x\end{bmatrix}f(x)$$

tenglikni o`rinligi isbotlanadi. Demak, (2.63) ayniyat to`g’ri. Qolgan ayniyatlar ham shunga o`xshash isbotlanadi.

**1.5 lemma isbotlandi.**

**1.6 – Lemma.** Ushbu (2.126), $\rho > -min\{a-c, b-c\} - 1/p$ va $1 \le p < q \le \infty$ shartlar bajarilsin. Agar $x^{\rho} f(x) \in L_p(0,\infty)$ bo`lsa, u holda $x^{\frac{1}{p}-\frac{1}{q}-c+\rho} F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x) \in L_q(0,\infty)$ bo`lib, quyidagi

$$\left\| x^{\frac{1}{p}-\frac{1}{q}-c+\rho} F_{x\infty}\begin{bmatrix} a, b \\ c; \ x \end{bmatrix} f(x) \right\|_q \le C_4 \left\| x^{\rho} f(x) \right\|_p \qquad (2.135)$$

baho o`rinli bo`ladi, bu erda $C_4$ o`zgarmas son $a, b, c$ va $p$ larga bog'liq. 1.6 lemmaning isboti 1.5 lemmaga o`xshash isbotlanadi.

## II bob bo‘yicha xulosalar

Ikki o‘zgaruvchili gipergeometrik funksiyalar haqida umumiy tushunchalar va kasr tartibli integro- differensial operatorlarning asosiy xossalari. Kasr tartibli integral va differensial operatorlar kompozitsiyasi natijasida hosil bo‘lgan asosiy ayniyatlar kelturilgan shu bilan birga bu operatorlarning qo‘llanilishi haqida so‘z yuritilgan.

# III BOB. 4-O‘LCHOVLI GIPERGEOMETRIK FUNKSIYALAR UCHUN YOYISH FOMULALARINING ASOSIY NATIJALARI BAYONI

## 3.1-§ Ikki o‘zgaruvchili uchunchi tartibli Campe de Feriet $F^{2;1;1}_{1;1;1}[x,y]$ gipergeometrik funktsiyasining integral ifodasi

Ma'lumki, nazariy fizika va zamonaviy matematikaning ko'plab muammolari bir nechta murakkab o'zgaruvchilarning turli xil gipergeometrik funktsiyalarini o'rganishga olib keladi. Bularga, masalan; super torlar nazariyasi muammolari [8], Mellin-Barns tipidagi kontur integrallarining analitik davomi [9, 10] va algebraik geometriya [11].

Ko'p o'zgaruvchilarning gipergeometrik funktsiyalari kvant maydon nazariyasida Knijnik-Zamolodchikov tenglamasining yechimi sifatida paydo bo'ladi, maydon nazariyasida va Vess-Zumino-Vitten modelidagi korrelyatsiya funktsiyalarining harakatini tavsiflaydi. Drinfeld [12] monodromiya tenglamalaridan biri (Drinfeld assotsiatori) beshburchakli tenglamani qanoatlantirishini va butun sonlarda bir nechta argumentlarga ega gipergeometrik funksiyalarni hosil qiluvchi funksiya ekanligini tasdiqladi. Ushbu yondashuv gipergeometrik tipdagi maxsus funktsiyalarni Li algebralari va kvant guruhlarini tasvirlash nazariyasidagi dolzarb masalalar bilan, shuningdek, boshqa amaliy masalalar bilan bog'lash imkonini beradi [12 - 14]. Gipergeometrik funksiyalar degenerativ differensial tenglamalar uchun chegaraviy masalalarni yechishda ham qo'llaniladi. [16-17]. Gipergeometrik funksiyalar uchun ba'zi kengaytirish formulalari [19-22] da isbotlangan. Ushbu hisobot $F^{2;1;1}_{1;1;1}[x,y]$ gipergeometrik funksiya uchun ba'zi integral tasvirlarni isbotlaydi [7, 18].

$$F^{2;1;1}_{1;1;1}\left[\begin{matrix} a_1, a_2; b; c; \\ d; e; f; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(b)_m(c)_n}{(d)_{m+n}(e)_m(f)_n\, m!n!} x^m y^n, \tag{3.1}$$

bunda $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)...(a+k-1),\ \ a \in \mathbb{C},\ k \in N_0 \equiv \{0,1,...\}$ - Poxgammer belgilashlari [7, 15, 18]. Quyidagi integral ko'rinishlar amalga oshiriladi:

$$F_{1;1;1}^{2;1;1}\left[\begin{matrix} a_1,a_2; & b; c; \\ d_1+d_2; e; f; \end{matrix} x,y\right]=\frac{\Gamma(d_1+d_2)}{\Gamma(d_1)\Gamma(d_2)}\times$$
$$\times\int_0^1 \xi^{d_1-1}(1-\xi)^{d_2-1} F_{1;1;1}^{2;1;1}\left[\begin{matrix} a_1,a_2; & b; & c; \\ -; d_1, e; d_2, f; \end{matrix} x\xi, y(1-\xi)\right]d\xi,\ \operatorname{Re} d_1>0,\ \operatorname{Re} d_2>0, \tag{3.2}$$

$$F_{1;1;1}^{2;1;1}\left[\begin{matrix} a_1,a_2; & b; c; \\ b+c; e; f; \end{matrix} x,y\right]=\frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\times$$
$$\times\int_0^1 \xi^{b-1}(1-\xi)^{c-1} F_4\left(a_1,a_2;e,f;x\xi,y(1-\xi)\right)d\xi,\ \operatorname{Re} b>0,\ \operatorname{Re} c>0, \tag{3.3}$$

$$F_{1;1;1}^{2;1;1}\left[\begin{matrix} a_1,a_2; b; c; \\ d; e; f; \end{matrix} x,y\right]=\frac{\Gamma(e)\Gamma(f)}{\Gamma(b)\Gamma(c)\Gamma(e-b)\Gamma(f-c)}\times$$
$$\times\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}(1-\xi)^{e-b-1}(1-\eta)^{f-c-1} F\left(a_1,a_2;d;x\xi+y\eta\right)d\xi d\eta, \tag{3.4}$$
$\operatorname{Re} e>\operatorname{Re} b>0,\ \operatorname{Re} f>\operatorname{Re} c>0,$

$$F_{1;1;1}^{2;1;1}\left[\begin{matrix} a_1,a_2; b; c; \\ d; e; f; \end{matrix} x,y\right]=\frac{\Gamma(d)}{\Gamma(a_1)\Gamma(d-a_1)}\times$$
$$\times\int_0^1 \xi^{a_1-1}(1-\xi)^{d-a_1-1} F_2\left(a_2;b,c;e,f;x\xi,y\xi\right)d\xi,\ \operatorname{Re} d>\operatorname{Re} a_1>0. \tag{3.5}$$

Maxsus funksiyalar nazariyasi matematik tahlilning bir sohasi sifatida, yuqori transsendental funktsiyalarni o'rganishda uzoq tarixga va boy mazmunga ega hamda funktsiyalar nazariyasida, integral va differentsial tenglamalar, matematikaning boshqa sohalarini o‘rganishda muhim ahamiyat kasb etadi.

Bir o‘zgaruvchili gipergeometrik funksiyalar nazariyasini o‘rganishda erishilgan ulkan muvaffaqiyatlar ikki va undan ko‘p o‘zgaruvchili gipergeometrik funksiyalar nazariyasining rivojlanishiga turtki bo‘ldi. 1889 yilda Gorn ikki o‘zgaruvchili gipergeometrik funksiyaning umumiy ta’rifini berdi, ularni to‘la va konflyuent funksiyalar guruhlariga ajratdi va 34 ta ikki o‘zgaruvchili funksiyalardan tuzilgan ro‘yxatni taqdim etdi.

Ikki o‘zgaruvchili gipergeometrik funksiyalarni chuqurroq o‘rganish maqsadida, birinchi bo‘lib, Berchnell va Chendi maxsus operatorlarni kiritib, ular yordamida Gorn ro‘yxatiga kirgan gipergeometrik funksiyalardan ba’zilari uchun yoyish formulalarini topdilar.Quyidagi belgilashlarni kiritamiz:

$a:=(a_1,...,a_2),\ b:=(b_1,...,b_2),\ c:=(c_1,...,c_2);\ |k|:=k_1+...k_n,\ k_1\geq 0,...k_n\geq 0;$

$x:=(x_1,...,x_2),\ y:=(y_1,...,y_p);\ |I|:=l_1+...l_n,\ l_1\geq 0,...l_n\geq 0;\ N:=\{1,2,3,\dots\};$

$(\lambda)_\nu -$ Poxgammer belgisi: $(\lambda)_0=1,\ (\lambda)_\nu=\lambda(\lambda+1)...(\lambda+\nu-1),\ \nu\in N.$

Gauss gipergeometrik funksiyasi quyidagi qator bilan ifodalanadi.

$$F(a;b;c;z)\equiv F\begin{vmatrix} a; & b; \\ c; & z \end{vmatrix}=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!},\ c\neq 0,-1,-2,...,|z|<1,$$

bu yerda a,b,c parametrlar z o‘zgaruvchiga bog’liq emas va ular kompleks qiymatlar qabul qilishi mumkin.

$F(a;b;c;z)$ Gauss gipergeometrik funksiyasining n ta kompleks o‘zgaruvchilar va ularga mos kelgan kompleks parametrlar holiga umumlashmalari bo‘lgan Laurichella gipergeometrik funksiyalarni qaraymiz:

$$F_A^{(n)}(a;b;c;x)=\sum_{|k|=0}^{\infty}\frac{(a)_{|k|}(b_1)_{k_1}...(b_n)_{k_n}}{(c_1)_{k_1}...(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}...\frac{x_1^{k_1}}{k_n!},\ \sum_{i=1}^{n}|x_i|<1;$$

$$F_B^{(n)}(a;b;c;x)=\sum_{|k|=0}^{\infty}\frac{(a_1)_{k_1}...(a_n)_{k_n}(b_1)_{k_1}...(b_n)_{k_n}}{(c)_{|k|}}\frac{x_1^{k_1}}{k_1!}...\frac{x_1^{k_1}}{k_n!},\ |x_1|<1,...|x_n|<1;$$

$$F_C^{(n)}(a;b;c;x)=\sum_{|k|=0}^{\infty}\frac{(a)_{|k|}(b)_{|k|}}{(c_1)_{k_1}...(c_n)_{k_n}}\frac{x_1^{k_1}}{k_1!}...\frac{x_1^{k_1}}{k_n!},\ \sum_{i=1}^{\infty}\sqrt{|x_1|}<1;$$

$$F_D^{(n)}(a;b;c;x)=\sum_{|k|=0}^{\infty}\frac{(a)_{|k|}(b_1)_{k_1}...(b_n)_{k_n}}{(c)_{|k|}}\frac{x_1^{k_1}}{k_1!}...\frac{x_1^{k_1}}{k_n!},\ |x_1|<1,...|x_n|<1.$$

2006 yilda A. Xasanov va X.M. Srivastavaning ishlarida Bechell- Chendi operatorlarini umumlashtiruvchi operatorlar kiritilgan, bir qator gipergeometrik funksiyalar, shu jumladan, uch o‘zgaruvchili Laurechella funksiyalari uchun yoyish formulalari isbotlangan.

### 3.2-§ To‘rt o‘zgaruvchili Gauss tipidagi gipergeometrik funksiya

Quyidagi to‘rt o‘zgaruvchili Gauss tipidagi gipergeometrik funksiyani urganamiz

$$F_{20}^{(4)}\left(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\, m!n!p!q!}x^m y^n z^p t^q$$

(3.6)

(3.6) munosabatdan quyidagini aniqlaymiz

$$\begin{aligned}
A_{m,n,p,q}&=\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\, m!n!p!q!}\\
A_{m+1,n,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+1}(b_2)_p(b_3)_q}{(c_1)_{m+q+1}(c_2)_n(c_3)_p\,(m+1)!n!p!q!}\\
A_{m,n+1,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+1}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_{n+1}(c_3)_p\, m!(n+1)!p!q!}\\
A_{m,n,p+1,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n}(b_2)_{p+1}(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_{p+1}\, m!n!(p+1)!q!}\\
A_{m,n,p,q+1}&=\frac{(a_1)_{m+n+p}(a_2)_{q+1}(b_1)_{m+n}(b_2)_p(b_3)_{q+1}}{(c_1)_{m+q+1}(c_2)_n(c_3)_p\, m!n!p!(q+1)!}
\end{aligned}\qquad(3.7)$$

Bundan esa

$$\begin{aligned}
\frac{A_{m+1,n,p,q}}{A_{m,n,p,q}}&=\frac{(c_1)_{m+q}}{(a_1)_{m+n+p}(b_1)_{m+n}}\frac{(a_1)_{m+n+p+1}(b_1)_{m+n+1}}{(c_1)_{m+q+1}(m+1)}\\
\frac{A_{m,n+1,p,q}}{A_{m,n,p,q}}&=\frac{(c_2)_n}{(a_1)_{m+n+p}(b_1)_{m+n}}\frac{(a_1)_{m+n+p+1}(b_1)_{m+n+1}}{(c_2)_{n+1}(n+1)}\\
\frac{A_{m,n,p+1,q}}{A_{m,n,p,q}}&=\frac{(c_3)_p}{(a_1)_{m+n+p}(b_2)_p}\frac{(a_1)_{m+n+p+1}(b_2)_{p+1}}{(c_3)_{p+1}(p+1)}\\
\frac{A_{m,n,p,q+1}}{A_{m,n,p,q}}&=\frac{(c_1)_{m+q}}{(a_2)_q(b_3)_q}\frac{(a_2)_{q+1}(b_3)_{q+1}}{(c_1)_{m+q+1}(q+1)}
\end{aligned}\qquad(3.8)$$

(3.8) tenglikni hosil qilamiz. Ba’zi soddalashtirishlardan so‘ng

$$
\begin{aligned}
&\frac{A_{m+1,n,p,q}}{A_{m,n,p,q}}=\frac{(a_1+m+n+p)(b_1+m+n)}{(c_1+m+q)(m+1)}\\
&\frac{A_{m,n+1,p,q}}{A_{m,n,p,q}}=\frac{(a_1+m+n+p)(b_1+m+n)}{(c_2+n)(n+1)}\\
&\frac{A_{m,n,p+1,q}}{A_{m,n,p,q}}=\frac{(a_1+m+n+p)(b_2+p)}{(c_3+p)(p+1)}\\
&\frac{A_{m,n,p,q+1}}{A_{m,n,p,q}}=\frac{(a_2+q)(b_3+q)}{(c_1+m+q)(q+1)}
\end{aligned}
\tag{3.9}
$$

(3.9) ifodaga kelamiz. Umumiy nazariyaga asosan (4) tenglikdan quyidagi 2-tartibli xususiy xosilali differensial tenglamalar sistemasi kelib chiqadi

$$
\begin{aligned}
&\left(c_1+x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u-\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u=0\\
&\left(c_2+y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial y}+1\right)y^{-1}u-\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u=0\\
&\left(c_3+z\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z}+1\right)z^{-1}u-\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_2+z\frac{\partial}{\partial z}\right)u=0\\
&\left(c_1+x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t}+1\right)t^{-1}u-\left(a_2+t\frac{\partial}{\partial t}\right)\left(b_3+t\frac{\partial}{\partial t}\right)u=0
\end{aligned}
\tag{3.10}
$$

(3.10) tenglamalar sistemasini (3.6) funksiya qanoatlantirishini ko’rsataylik. (3.10) sistemani birinchi tenglamasiga (3.6) funksiyani qo‘yamiz va quyidagi tenglikliklarni e’tiborga olsak,

$$
\begin{aligned}
&x^{-1}u=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}x^{m-1}y^nz^pt^q,\\
&u=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}x^my^nz^pt^q,\\
&x\frac{\partial}{\partial x}x^m=mx^m,\ y\frac{\partial}{\partial y}y^n=ny^n,\ z\frac{\partial}{\partial z}z^p=pz^p,\ t\frac{\partial}{\partial t}t^q=qt^q
\end{aligned}
\tag{3.11}
$$

va

$$\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}\left(x\frac{\partial}{\partial x}x^{m-1}y^nz^pt^q+x^{m-1}y^nz^pt^q\right),$$

$$\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u= \tag{3.12}$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}\left(b_1x^my^nz^pt^q+x\frac{\partial}{\partial x}x^my^nz^pt^q+y\frac{\partial}{\partial y}x^my^nz^pt^q\right)$$

o‘rinli. Bundan quyidagi tengliklarni hosil qilamiz

$$\left(c_1+x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u=$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}\left(c_1m+m(m-1)+mq\right)x^{m-1}y^nz^pt^q,$$

$$\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u= \tag{3.13}$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}(b_1+m+n)(a_1+m+n+p)x^my^nz^pt^q.$$

(3.13) tenglikni (3.12) sistemaning birinchi tenglamasiga qo‘ysak

$$\left(c_1+x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u-\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u=$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+1+n+p}(a_2)_q(b_1)_{m+1+n}(b_2)_p(b_3)_q}{(c_1)_{m+1+q}(c_2)_n(c_3)_p(m+1)m!n!p!q!}(c_1+m+q)(m+1)x^my^nz^pt^q- \tag{3.14}$$

$$-\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}(b_1+m+n)(a_1+m+n+p)x^my^nz^pt^q.$$

Endi quyidagi tenglikni inobatga olsak

$$(a_1)_{m+1+n+p}=(a_1+m+n+p)(a_1)_{m+n+p},$$

$$(b_1)_{m+1+n}=(b_1+m+n)(b_1)_{m+n},\ (c_1)_{m+1+q}=(c_1+m+q)(c_1)_{m+q}, \tag{3.15}$$

(3.15) ifodadan

$$\left(c_1+x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}\right)\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u-\left(a_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(b_1+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u=$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p\,m!n!p!q!}\times \tag{3.16}$$

$$\times\left[\begin{array}{l}\dfrac{(a_1+m+n+p)(b_1+m+n)(c_1+m+q)(m+1)}{(c_1+m+q)(m+1)}x^my^nz^pt^q-\\ -(a_1+m+n+p)(b_1+m+n)x^my^nz^pt^q\end{array}\right]=0$$

kelib chiqadi. Shunday kilib, (3.16) funksiya (3.10) tenglamalar sistemasining birinchi tenglamasini qanoatlantirishini ko‘rsatdik. (3.10) sistemaning boshqa tenglamalari ham o‘rinli ekanligi xuddi shunday isbotlanadi.

Endi (3.10) sistemani soddalashtiramiz.

$$\left(c_1 + x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right)u_x - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_1 u + xu_x + yu_y\right) = 0$$
$$\left(c_2 + y\frac{\partial}{\partial y}\right)u_y - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_1 u + xu_x + yu_y\right) = 0$$
$$\left(c_3 + z\frac{\partial}{\partial z}\right)u_z - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_2 u + zu_z\right) = 0$$
$$\left(c_1 + x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right)u_t - \left(a_2 + t\frac{\partial}{\partial t}\right)\left(b_3 u + tu_t\right) = 0$$

$$c_1u_x + xu_{xx} + tu_{xt} - \begin{pmatrix} a_1\left(b_1u + xu_x + yu_y\right) + x\frac{\partial}{\partial x}\left(b_1u + xu_x + yu_y\right) \\ +y\frac{\partial}{\partial y}\left(b_1u + xu_x + yu_y\right) + z\frac{\partial}{\partial z}\left(b_1u + xu_x + yu_y\right) \end{pmatrix} = 0$$
$$c_2u_y + yu_{yy} - \begin{pmatrix} a_1\left(b_1u + xu_x + yu_y\right) + x\frac{\partial}{\partial x}\left(b_1u + xu_x + yu_y\right) \\ +y\frac{\partial}{\partial y}\left(b_1u + xu_x + yu_y\right) + z\frac{\partial}{\partial z}\left(b_1u + xu_x + yu_y\right) \end{pmatrix} = 0$$
$$c_3u_z + zu_{zz} - \begin{pmatrix} a_1\left(b_2u + zu_z\right) + x\frac{\partial}{\partial x}\left(b_2u + zu_z\right) \\ +y\frac{\partial}{\partial y}\left(b_2u + zu_z\right) + z\frac{\partial}{\partial z}\left(b_2u + zu_z\right) \end{pmatrix} = 0$$
$$c_1u_t + xu_{xt} + tu_{tt} - \left(a_2\left(b_3u + tu_t\right) + t\frac{\partial}{\partial t}\left(b_3u + tu_t\right)\right) = 0$$

$$c_1u_x + xu_{xx} + tu_{xt} - \begin{pmatrix} xxu_{xx} + 2xyu_{xy} + xzu_{xz} + yyu_{yy} + yzu_{yz} + \left(a_1 + b_1 + 1\right)xu_x \\ +\left(a_1 + b_1 + 1\right)yu_y + b_1zu_z + a_1b_1u \end{pmatrix} = 0$$
$$c_2u_y + yu_{yy} - \begin{pmatrix} yyu_{yy} + xxu_{xx} + 2xyu_{xy} + xzu_{xz} + yzu_{yz} + \left(a_1 + b_1 + 1\right)xu_x \\ +\left(a_1 + b_1 + 1\right)yu_y + b_1zu_z + a_1b_1u \end{pmatrix} = 0$$
$$c_3u_z + zu_{zz} - \left(zzu_{zz} + xzu_{xz} + yzu_{yz} + b_2xu_x + b_2yu_y + \left(a_1 + b_2 + 1\right)zu_z + a_1b_2u\right) = 0$$
$$c_1u_t + xu_{xt} + tu_{tt} - \left(ttu_{tt} + \left(a_2 + b_3 + 1\right)tu_t + a_2b_3u\right) = 0$$

$$x(1-x)u_{xx}-2xyu_{xy}-xzu_{xz}+tu_{xt}-y^2u_{yy}-yzu_{yz}+\left[c_1-(a_1+b_1+1)x\right]u_x$$
$$-(a_1+b_1+1)yu_y-b_1zu_z-a_1b_1u=0$$
$$y(1-y)u_{yy}-x^2u_{xx}-2xyu_{xy}-xzu_{xz}-yzu_{yz}-(a_1+b_1+1)xu_x$$
$$+\left[c_2-(a_1+b_1+1)y\right]u_y-b_1zu_z-a_1b_1u=0$$
$$z(1-z)u_{zz}-xzu_{xz}-yzu_{yz}-b_2xu_x-b_2yu_y+\left[c_3-(a_1+b_2+1)z\right]u_z-a_1b_2u=0$$
$$t(1-t)u_{tt}+xu_{xt}+\left[c_1-(a_2+b_3+1)t\right]u_t-a_2b_3u=0$$

Va nihoyat ikkinchi tartibli to'rt o'zgaruvchili xususiy xosilali differensial tenglamalar sistemasini hosil qilamiz

$$\begin{cases} x(1-x)u_{xx}-2xyu_{xy}-xzu_{xz}+tu_{xt}-y^2u_{yy}-yzu_{yz} \\ +\left[c_1-(a_1+b_1+1)x\right]u_x-(a_1+b_1+1)yu_y-b_1zu_z-a_1b_1u=0 \\ y(1-y)u_{yy}-x^2u_{xx}-2xyu_{xy}-xzu_{xz}-yzu_{yz} \\ -(a_1+b_1+1)xu_x+\left[c_2-(a_1+b_1+1)y\right]u_y-b_1zu_z-a_1b_1u=0 \\ z(1-z)u_{zz}-xzu_{xz}-yzu_{yz} \\ \qquad -b_2xu_x-b_2yu_y+\left[c_3-(a_1+b_2+1)z\right]u_z-a_1b_2u=0 \\ \qquad t(1-t)u_{tt}+xu_{xt}+\left[c_1-(a_2+b_3+1)t\right]u_t-a_2b_3u=0 \end{cases} \tag{3.18}$$

Endi esa, (5) sistema koordinatalar boshida qanday chiziqli erkli yechimlari bor ekanligini aniqlaymiz. (5) sistemani yechimlarini quyidagi ko'rinishda izlaymiz

$$u=x^\alpha y^\beta z^\gamma t^\delta w \tag{3.19}$$

Buning uchun quyidagi xususiy hosilalarni topamiz

$$u_x=\alpha x^{\alpha-1}y^\beta z^\gamma t^\delta w+x^\alpha y^\beta z^\gamma t^\delta w_x$$
$$u_y=\beta x^\alpha y^{\beta-1}z^\gamma t^\delta w+x^\alpha y^\beta z^\gamma t^\delta w_y$$
$$u_z=\gamma x^\alpha y^\beta z^{\gamma-1}t^\delta w+x^\alpha y^\beta z^\gamma t^\delta w_z$$
$$u_t=\delta x^\alpha y^\beta z^\gamma t^{\delta-1}w+x^\alpha y^\beta z^\gamma t^\delta w_t$$
$$u_{xx}=\alpha(\alpha-1)x^{\alpha-2}y^\beta z^\gamma t^\delta w+2\alpha x^{\alpha-1}y^\beta z^\gamma t^\delta w_x+x^\alpha y^\beta z^\gamma t^\delta w_{xx}$$
$$u_{yy}=\beta(\beta-1)x^\alpha y^{\beta-2}z^\gamma t^\delta w+2\beta x^\alpha y^{\beta-1}z^\gamma t^\delta w_y+x^\alpha y^\beta z^\gamma t^\delta w_{yy}$$
$$u_{zz}=\gamma(\gamma-1)x^\alpha y^\beta z^{\gamma-2}t^\delta w+2\gamma x^\alpha y^\beta z^{\gamma-1}t^\delta w_z+x^\alpha y^\beta z^\gamma t^\delta w_{zz}$$
$$u_{tt}=\delta(\delta-1)x^\alpha y^\beta z^\gamma t^{\delta-2}w+2\delta x^\alpha y^\beta z^\gamma t^{\delta-1}w_t+x^\alpha y^\beta z^\gamma t^\delta w_{tt}$$
$$u_{xy}=\alpha\beta x^{\alpha-1}y^{\beta-1}z^\gamma t^\delta w+\alpha x^{\alpha-1}y^\beta z^\gamma t^\delta w_y+\beta x^\alpha y^{\beta-1}z^\gamma t^\delta w_x+x^\alpha y^\beta z^\gamma t^\delta w_{xy}$$
$$u_{xz}=\alpha\gamma x^{\alpha-1}y^\beta z^{\gamma-1}t^\delta w+\alpha x^{\alpha-1}y^\beta z^\gamma t^\delta w_z+\gamma x^\alpha y^\beta z^{\gamma-1}t^\delta w_x+x^\alpha y^\beta z^\gamma t^\delta w_{xz}$$

$$u_{xt}=\alpha\delta x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta-1}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_t+\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xt}$$

$$u_{yz}=\beta\gamma x^{\alpha}y^{\beta-1}z^{\gamma-1}t^{\delta}w+\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_z+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_y+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{yz}$$

$$u_{yt}=\beta\delta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta-1}w+\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_t+\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_y+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{yt}$$

$$u_{zt}=\gamma\delta x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta-1}w+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_t+\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_z+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{zt}$$

1. Topilgan xususiy xosilalarni (3.10) sistemaning birinchi tenglamasiga kuyamiz, u holda

$$\begin{aligned}
&x(1-x)\left(\alpha(\alpha-1)x^{\alpha-2}y^{\beta}z^{\gamma}t^{\delta}w+2\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xx}\right)\\
&-2xy\left(\alpha\beta x^{\alpha-1}y^{\beta-1}z^{\gamma}t^{\delta}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_y+\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xy}\right)\\
&-xz\left(\alpha\gamma x^{\alpha-1}y^{\beta}z^{\gamma-1}t^{\delta}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_z+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xz}\right)\\
&+t\left(\alpha\delta x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta-1}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_t+\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xt}\right)\\
&-y^2\left(\beta(\beta-1)x^{\alpha}y^{\beta-2}z^{\gamma}t^{\delta}w+2\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_y+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{yy}\right)\\
&-yz\left(\beta\gamma x^{\alpha}y^{\beta-1}z^{\gamma-1}t^{\delta}w+\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_z+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_y+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{yz}\right)\\
&+\left[c_1-(a_1+b_1+1)x\right]\left(\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_x\right)\\
&-(a_1+b_1+1)y\left(\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_y\right)-b_1z\left(\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_z\right)\\
&-a_1b_1x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w=0
\end{aligned}$$

$$\begin{aligned}
&x(1-x)\left(\alpha(\alpha-1)x^{-2}w+2\alpha x^{-1}w_x+w_{xx}\right)-2xy\left(\alpha\beta x^{-1}y^{-1}w+\alpha x^{-1}w_y+\beta y^{-1}w_x+w_{xy}\right)\\
&-xz\left(\alpha\gamma x^{-1}z^{-1}w+\alpha x^{-1}w_z+\gamma z^{-1}w_x+w_{xz}\right)+t\left(\alpha\delta x^{-1}t^{-1}w+\alpha x^{-1}w_t+\delta t^{-1}w_x+w_{xt}\right)\\
&-y^2\left(\beta(\beta-1)y^{-2}w+2\beta y^{-1}w_y+w_{yy}\right)-yz\left(\beta\gamma y^{-1}z^{-1}w+\beta y^{-1}w_z+\gamma z^{-1}t^{\delta}w_y+w_{yz}\right)\\
&+\left[c_1-(a_1+b_1+1)x\right]\left(\alpha x^{-1}w+w_x\right)-(a_1+b_1+1)y\left(\beta y^{-1}w+w_y\right)-b_1z\left(\gamma z^{-1}w+w_z\right)\\
&-a_1b_1w=0
\end{aligned}$$

$$\begin{aligned}
&x(1-x)w_{xx}-2xyw_{xy}-xzw_{xz}+tw_{xt}-y^2w_{yy}-yzw_{yz}+\left[\delta+c_1+2\alpha-(2\alpha+2\beta+\gamma+a_1+b_1+1)x\right]w_x\\
&-(2\alpha+2\beta+\gamma+a_1+b_1+1)yw_y-(\alpha+\beta+b_1)zw_z+\alpha x^{-1}tw_t\\
&-\left\{\begin{aligned}&-\alpha(\alpha-1+\delta+c_1)x^{-1}\\&+\alpha(\alpha-1)+2\alpha\beta+\alpha\gamma+\beta(\beta-1)+\beta\gamma\\&+(a_1+b_1+1)\alpha+(a_1+b_1+1)\beta+b_1\gamma+a_1b_1\end{aligned}\right\}w=0
\end{aligned}$$

$$\begin{aligned}
&x(1-x)w_{xx}-2xyw_{xy}-xzw_{xz}+tw_{xt}-y^2w_{yy}-yzw_{yz}+\\
&+\left\{\delta+c_1+2\alpha-\left[(\alpha+\beta+\gamma+a_1)+(\alpha+\beta+b_1)+1\right]x\right\}w_x\\
&-\left[(\alpha+\beta+\gamma+a_1)+(\alpha+\beta+b_1)+1\right]yw_y-(\alpha+\beta+b_1)zw_z+\alpha x^{-1}tw_t\\
&-\left\{-\alpha(\alpha-1+\delta+c_1)x^{-1}+(\alpha+\beta+\gamma+a_1)(\alpha+\beta+b_1)\right\}w=0
\end{aligned}$$

Demak $\alpha = 0$, bo'lishi kerak. Bundan esa

$$
\begin{aligned}
& x(1-x)w_{xx} - 2xyw_{xy} - xzw_{xz} + tw_{xt} - y^2 w_{yy} - yzw_{yz} + \\
& + \left[\delta + c_1 - \left(\beta + \gamma + a_1 + (\beta + b_1) + 1\right)x\right] w_x - \\
& - \left[\beta + \gamma + a_1 + (\beta + b_1) + 1\right] yw_y - (\beta + b_1) zw_z - \\
& - \left\{\beta(\beta - 1) + \beta\gamma + (a_1 + b_1 + 1)\beta + b_1\gamma + a_1 b_1\right\} w = 0.
\end{aligned} \tag{3.20}
$$

2. Topilgan xususiy hosilalarni (3.10) sistemaning ikkinchi tenglamasiga qo‘yamiz, u holda

$$
\begin{aligned}
& y(1-y)\left(\beta(\beta-1)x^\alpha y^{\beta-2} z^\gamma t^\delta w + 2\beta x^\alpha y^{\beta-1} z^\gamma t^\delta w_y + x^\alpha y^\beta z^\gamma t^\delta w_{yy}\right) \\
& -x^2\left(\alpha(\alpha-1)x^{\alpha-2} y^\beta z^\gamma t^\delta w + 2\alpha x^{\alpha-1} y^\beta z^\gamma t^\delta w_x + x^\alpha y^\beta z^\gamma t^\delta w_{xx}\right) \\
& -2xy\left(\alpha\beta x^{\alpha-1} y^{\beta-1} z^\gamma t^\delta w + \alpha x^{\alpha-1} y^\beta z^\gamma t^\delta w_y + \beta x^\alpha y^{\beta-1} z^\gamma t^\delta w_x + x^\alpha y^\beta z^\gamma t^\delta w_{xy}\right) \\
& -xz\left(\alpha\gamma x^{\alpha-1} y^\beta z^{\gamma-1} t^\delta w + \alpha x^{\alpha-1} y^\beta z^\gamma t^\delta w_z + \gamma x^\alpha y^\beta z^{\gamma-1} t^\delta w_x + x^\alpha y^\beta z^\gamma t^\delta w_{xz}\right) \\
& -yz\left(\beta\gamma x^\alpha y^{\beta-1} z^{\gamma-1} t^\delta w + \beta x^\alpha y^{\beta-1} z^\gamma t^\delta w_z + \gamma x^\alpha y^\beta z^{\gamma-1} t^\delta w_y + x^\alpha y^\beta z^\gamma t^\delta w_{yz}\right) \\
& -(a_1 + b_1 + 1)x\left(\alpha x^{\alpha-1} y^\beta z^\gamma t^\delta w + x^\alpha y^\beta z^\gamma t^\delta w_x\right) + \left[c_2 - (a_1 + b_1 + 1)y\right]\left(\beta x^\alpha y^{\beta-1} z^\gamma t^\delta w + x^\alpha y^\beta z^\gamma t^\delta w_y\right) \\
& -b_1 z\left(\gamma x^\alpha y^\beta z^{\gamma-1} t^\delta w + x^\alpha y^\beta z^\gamma t^\delta w_z\right) - a_1 b_1 x^\alpha y^\beta z^\gamma t^\delta w = 0
\end{aligned}
$$

$$
\begin{aligned}
& y(1-y)\left(\beta(\beta-1)y^{-2}w + 2\beta y^{-1}w_y + w_{yy}\right) - x^2\left(\alpha(\alpha-1)x^{-2}w + 2\alpha x^{-1}w_x + w_{xx}\right) \\
& -2xy\left(\alpha\beta x^{-1}y^{-1}w + \alpha x^{-1}w_y + \beta y^{-1}w_x + w_{xy}\right) - xz\left(\alpha\gamma x^{-1}z^{-1}w + \alpha x^{-1}w_z + \gamma z^{-1}w_x + w_{xz}\right) \\
& -yz\left(\beta\gamma y^{-1}z^{-1}w + \beta y^{-1}w_z + \gamma z^{-1}w_y + w_{yz}\right) - (a_1 + b_1 + 1)x\left(\alpha x^{-1}w + w_x\right) \\
& + \left[c_2 - (a_1 + b_1 + 1)y\right]\left(\beta y^{-1}w + w_y\right) - b_1 z\left(\gamma z^{-1}w + w_z\right) - a_1 b_1 w = 0
\end{aligned}
$$

$$
\begin{aligned}
& y(1-y)w_{yy} - x^2 w_{xx} - 2xyw_{xy} - xzw_{xz} - yzw_{yz} - 2\alpha xw_x - 2\beta xw_x - \gamma xw_x - (a_1 + b_1 + 1)xw_x \\
& +2\beta(1-y)w_y - 2\alpha yw_y - \gamma yw_y + c_2 w_y - (a_1 + b_1 + 1)yw_y - \alpha zw_z - \beta zw_z - b_1 zw_z \\
& +\beta(\beta-1)(1-y)y^{-1}w - \alpha(\alpha-1)w - 2\alpha\beta w - \alpha\gamma w - \beta\gamma w \\
& -(a_1 + b_1 + 1)\alpha w + c_2\beta y^{-1}w - (a_1 + b_1 + 1)\beta w - b_1\gamma zz^{-1}w - a_1 b_1 w = 0
\end{aligned}
$$

$$
\begin{aligned}
& y(1-y)w_{yy} - x^2 w_{xx} - 2xyw_{xy} - xzw_{xz} - yzw_{yz} - (2\alpha + 2\beta + \gamma + a_1 + b_1 + 1)xw_x \\
& + \left[2\beta + c_2 - (2\alpha + 2\beta + \gamma + a_1 + b_1 + 1)y\right]w_y - (\alpha + \beta + b_1)zw_z \\
& - \left\{\begin{matrix} -\beta(\beta - 1 + c_2)y^{-1} + \beta(\beta-1) + \alpha(\alpha-1) + 2\alpha\beta + \alpha\gamma + \beta\gamma \\ + (a_1 + b_1 + 1)\alpha + (a_1 + b_1 + 1)\beta + b_1\gamma + a_1 b_1 \end{matrix}\right\} w = 0
\end{aligned}
$$

$$
\begin{aligned}
& y(1-y)w_{yy} - x^2 w_{xx} - 2xyw_{xy} - xzw_{xz} - yzw_{yz} - \\
& - \left[(\alpha + \beta + \gamma + a_1) + (\alpha + \beta + b_1) + 1\right]xw_x + \\
& + \left\{2\beta + c_2 - \left[(\alpha + \beta + \gamma + a_1) + (\alpha + \beta + b_1) + 1\right]y\right\}w_y - (\alpha + \beta + b_1)zw_z - \\
& - \left\{-\beta(\beta - 1 + c_2)y^{-1} + (\alpha + \beta + \gamma + a_1)(\alpha + \beta + b_1)\right\}w = 0.
\end{aligned} \tag{3.21}
$$

xosil qilamiz.

3. Topilgan xususiy hosilalarni (3.10) sistemaning uchunchi tenglamasiga qo'yamiz, u holda

$$\begin{aligned}
&z(1-z)\left(\gamma(\gamma-1)x^{\alpha}y^{\beta}z^{\gamma-2}t^{\delta}w+2\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_z+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{zz}\right)\\
&-xz\left(\alpha\gamma x^{\alpha-1}y^{\beta}z^{\gamma-1}t^{\delta}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_z+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xz}\right)\\
&-yz\left(\beta\gamma x^{\alpha}y^{\beta-1}z^{\gamma-1}t^{\delta}w+\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w_z+\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w_y+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{yz}\right)\\
&-b_2x\left(\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_x\right)-b_2y\left(\beta x^{\alpha}y^{\beta-1}z^{\gamma}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_y\right)\\
&+\left[c_3-(a_1+b_2+1)z\right]\left(\gamma x^{\alpha}y^{\beta}z^{\gamma-1}t^{\delta}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_z\right)-a_1b_2x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w=0
\end{aligned}$$

$$\begin{aligned}
&z(1-z)\left(\gamma(\gamma-1)z^{-2}w+2\gamma z^{-1}w_z+w_{zz}\right)-xz\left(\alpha\gamma x^{-1}z^{-1}w+\alpha x^{-1}w_z+\gamma z^{-1}w_x+w_{xz}\right)\\
&-yz\left(\beta\gamma y^{-1}z^{-1}w+\beta y^{-1}w_z+\gamma z^{-1}w_y+w_{yz}\right)-b_2x\left(\alpha x^{-1}w+w_x\right)-b_2y\left(\beta y^{-1}w+w_y\right)\\
&+\left[c_3-(a_1+b_2+1)z\right]\left(\gamma z^{-1}w+w_z\right)-a_1b_2w=0
\end{aligned}$$

$$\begin{aligned}
&z(1-z)w_{zz}-xzw_{xz}-yzw_{yz}-(\gamma+b_2)xw_x\\
&-(\gamma+b_2)yw_y+\left\{2\gamma+c_3-\left[\gamma+\alpha+\beta+a_1+(\gamma+b_2)+1\right]z\right\}w_z\\
&-\left\{\begin{array}{l}-\gamma(\gamma-1+c_3)z^{-1}\\ +\gamma(\gamma-1)+\alpha\gamma+\beta\gamma+b_2\alpha+b_2\beta+(a_1+b_2+1)\gamma+a_1b_2\end{array}\right\}w=0
\end{aligned}$$

$$\begin{aligned}
&z(1-z)w_{zz}-xzw_{xz}-yzw_{yz}-(\gamma+b_2)xw_x-(\gamma+b_2)yw_y+\\
&+\left\{2\gamma+c_3-\left[(\alpha+\beta+\gamma+a_1)+(\gamma+b_2)+1\right]z\right\}w_z-\\
&-\left[-\gamma(\gamma-1+c_3)z^{-1}+(\alpha+\beta+\gamma+a_1)+(\gamma+b_2)\right]w=0.
\end{aligned} \qquad (3.22)$$

munosabatni hosil qilamiz.

4. Topilgan xususiy hosilalarni (3.10) sistemaning to'rtinchi tenglamasiga qo'yamiz, u holda

$$\begin{aligned}
&t(1-t)\left(\delta(\delta-1)x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-2}w+2\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_t+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{tt}\right)\\
&+x\left(\alpha\delta x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta-1}w+\alpha x^{\alpha-1}y^{\beta}z^{\gamma}t^{\delta}w_t+\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w_x+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_{xt}\right)\\
&+\left[c_1-(a_2+b_3+1)t\right]\left(\delta x^{\alpha}y^{\beta}z^{\gamma}t^{\delta-1}w+x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w_t\right)-a_2b_3x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w=0
\end{aligned}$$

$$\begin{aligned}
&t(1-t)\left(\delta(\delta-1)t^{-2}w+2\delta t^{-1}w_t+w_{tt}\right)+x\left(\alpha\delta x^{-1}t^{-1}w+\alpha x^{-1}w_t+\delta t^{-1}w_x+w_{xt}\right)\\
&+\left[c_1-(a_2+b_3+1)t\right]\left(\delta t^{-1}w+w_t\right)-a_2b_3w=0
\end{aligned}$$

$$\begin{aligned}
&t(1-t)w_{tt}+xw_{xt}+\delta xt^{-1}w_x+2\delta(1-t)w_t+\alpha w_t+c_1w_t-(a_2+b_3+1)tw_t\\
&+\delta(\delta-1)t^{-1}(1-t)w+\alpha\delta t^{-1}w+c_1\delta t^{-1}w-(a_2+b_3+1)\delta w-a_2b_3w=0
\end{aligned}$$

$$t\left(1-t\right)w_{tt}+xw_{xt}+\delta xt^{-1}w_x+\left\{2\delta+\alpha+c_1-\left[2\delta+a_2+b_3+1\right]t\right\}w_t$$
$$-\left\{-\delta\left(\delta-1+\alpha+c_1\right)t^{-1}+\delta\left(\delta-1\right)+\left(a_2+b_3+1\right)\delta+a_2b_3\right\}w=0 \tag{3.23}$$

Demak $\delta=0$, bo'lishi kerak. Shunday qilib, (3.20)-(3.23) teglamalarni birlashtirib quyidagi sistemani hosil qilamiz

$$\begin{cases} x\left(1-x\right)w_{xx}-2xyw_{xy}-xzw_{xz}+tw_{xt}-y^2w_{yy}-yzw_{yz}+ \\ \qquad +\left\{c_1-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]x\right\}w_x- \\ -\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]yw_y-\left(\beta+b_1\right)zw_z \\ \qquad -\left\{\beta^2+\left(\gamma+a_1+b_1\right)\beta+b_1\gamma+a_1b_1\right\}w=0, \\ y\left(1-y\right)w_{yy}-x^2w_{xx}-2xyw_{xy}-xzw_{xz}-yzw_{yz}-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]xw_x+ \\ +\left\{2\beta+c_2-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]y\right\}w_y-\left(\beta+b_1\right)zw_z- \\ -\left\{-\beta\left(\beta-1+c_2\right)y^{-1}+\beta^2+\left(\gamma+a_1+b_1\right)\beta+b_1\gamma+a_1b_1\right\}w=0, \\ z\left(1-z\right)w_{zz}-xzw_{xz}-yzw_{yz}-\left(\gamma+b_2\right)xw_x- \\ \qquad -\left(\gamma+b_2\right)yw_y+\left\{2\gamma+c_3-\left[\left(\gamma+\beta+a_1\right)+\left(\gamma+b_2\right)+1\right]z\right\}w_z- \\ -\left\{-\gamma\left(\gamma-1+c_3\right)z^{-1}+\gamma^2+\left(\beta+a_1+b_2\right)\gamma+b_2\beta+a_1b_2\right\}w=0, \\ \qquad t\left(1-t\right)w_{tt}+xw_{xt}+\left[c_1-\left(a_2+b_3+1\right)t\right]w_t-a_2b_3w=0. \end{cases} \tag{3.24}$$

Quyidagi tengliklardan foydalansak

$$\beta^2+\left(\gamma+a_1+b_1\right)\beta+\gamma b_1+a_1b_1=\left(\beta+\gamma+a_1\right)\left(\beta+b_1\right)$$

$$\gamma^2+\left(\beta+a_1+b_2\right)\gamma+\beta b_2+a_1b_2=\left(\gamma+\beta+a_1\right)\left(\gamma+b_2\right)$$

(3.24) sistemaning ko'rinishi quyidagicha bo'ladi

$$\begin{cases} x\left(1-x\right)w_{xx}-2xyw_{xy}-xzw_{xz}+tw_{xt}-y^2w_{yy}-yzw_{yz}+ \\ +\left\{c_1-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]x\right\}w_x-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]yw_y- \\ \qquad -\left(\beta+b_1\right)zw_z-\left(\beta+\gamma+a_1\right)\left(\beta+b_1\right)w=0, \\ y\left(1-y\right)w_{yy}-x^2w_{xx}-2xyw_{xy}-xzw_{xz}-yzw_{yz}-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]xw_x+ \\ +\left\{2\beta+c_2-\left[\left(\beta+\gamma+a_1\right)+\left(\beta+b_1\right)+1\right]y\right\}w_y-\left(\beta+b_1\right)zw_z- \\ \qquad -\left[-\beta\left(\beta-1+c_2\right)y^{-1}+\left(\beta+\gamma+a_1\right)\left(\beta+b_1\right)\right]w=0, \\ z\left(1-z\right)w_{zz}-xzw_{xz}-yzw_{yz}-\left(\gamma+b_2\right)xw_x-\left(\gamma+b_2\right)yw_y+ \\ \qquad +\left\{2\gamma+c_3-\left[\left(\gamma+\beta+a_1\right)+\left(\gamma+b_2\right)+1\right]z\right\}w_z- \\ \qquad -\left[-\gamma\left(\gamma-1+c_3\right)z^{-1}+\left(\gamma+\beta+a_1\right)+\left(\gamma+b_2\right)\right]w=0, \\ \qquad t\left(1-t\right)w_{tt}+xw_{xt}+\left[c_1-\left(a_2+b_3+1\right)t\right]w_t-a_2b_3w=0. \end{cases} \tag{3.25}$$

Demak, quyidagi sistema o'rinli bo'lishi kerak

$$\begin{cases} \alpha = 0 \\ \beta(\beta - 1 + c_2) = 0 \\ \gamma(\gamma - 1 + c_3) = 0 \\ \delta = 0 \end{cases} \qquad (3.26)$$

(3.26) sistemaning yechimlari esa, quyidagicha

$$\begin{array}{lcccc} & 1 & 2 & 3 & 4 \\ \alpha := & 0 & 0 & 0 & 0 \\ \beta := & 0 & 1-c_2 & & 1-c_2 \\ \gamma := & 0 & 0 & 1-c_3 & 1-c_3 \\ \delta := & 0 & 0 & 0 & 0 \end{array} \qquad (3.27)$$

(3.27) yechimdan foydalanib, (3.10) sistemaning koordinata boshidagi chiziqli erkli yechimlarini topamiz

1. $$\begin{array}{l} \alpha = \beta = \gamma = \delta = 0, \\ u_1(x,y,z,t) = F_{20}^{(4)}(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t), \end{array} \qquad (3.28)$$

2.

$$\begin{array}{l} \alpha = \gamma = \delta = 0,\ \beta = 1-c_2, \\ u_2(x,y,z,t) = y^{1-c_2} F_{20}^{(4)}(1-c_2+a_1,a_2,1-c_2+b_1,b_2,b_3;c_1,2-c_2,c_3;x,y,z,t), \end{array} \qquad (3.29)$$

3.

$$\begin{array}{l} \alpha = \beta = \delta = 0,\ \gamma = 1-c_3, \\ u_3(x,y,z,t) = z^{1-c_3} F_{20}^{(4)}(1-c_3+a_1,a_2,b_1,1-c_3+b_2,b_3;c_1,c_2,2-c_3;x,y,z,t), \end{array} \qquad (3.30)$$

4.

$$\begin{array}{l} \alpha = \delta = 0,\ \beta = 1-c_2,\ \gamma = 1-c_3, \\ u_4(x,y,z,t) = y^{1-c_2} z^{1-c_3} \times \\ \times F_{20}^{(4)}(2-c_2-c_3+a_1,a_2,1-c_2+b_1,1-c_3+b_2,b_3;c_1,2-c_2,2-c_3;x,y,z,t). \end{array} \qquad (3.31)$$

## III bob bo‘yicha xulosalar

Ikki o‘zgaruvchili gipergeometrik funksiyalarni chuqurroq o‘rganish maqsadida, birinchi bo‘lib, Berchnell va Chendi maxsus operatorlarni kiritib, ular yordamida Gorn ro‘yxatiga kirgan gipergeometrik funksiyalardan ba’zilari uchun yoyish formulalari keltirilgan. Gipergeometrik funksiyalar uchun ba’zi kengaytirish formulalari keltirilgan. Ushbu $F_{1;1;1}^{2;1;1}\left[x, y\right]$ gipergeometrik funksiya uchun ba’zi integral tasvirlar keltirilgan.

## XULOSA

Bizga ma'lumki, ko'p o'zgaruvchili gipergeometrik funksiyalarni tadqiq etishda maxsus funksiyalar muhim ahamiyat kasb etadi, shu sababali mazkur bobda gamma va beta funksiyalar, Gaussning gipergeometrik funksiyasi haqida tushunchalar, Gauss tenglamasining Kimmer yechimlari, ularning integral ifodalari, gipergeometrik funksiylarni analitik davom ettirish va ular orasidagi asosiy funksional bog'lanishlar haqida bilish juda muhim.

Shu bilan birga ikki o'zgaruvchili gipergeometrik funksiyalar haqida umumiy tushunchalar va kasr tartibli integro- differensial operatorlarning asosiy xossalari, kasr tartibli integral va differensial operatorlar kompozitsiyasi natijasida hosil bo'lgan asosiy ayniyatlarning qo'llanilishiga doir masalalarni hal qilish muhim ahamiyat kasb etadi.

Oxirgi yillarda gipergeometrik funksiyalar va ularni buziladigan elliptic tenglamalar uchun chegaraviy masalalarni yechishga qo'llash bo'yicha olib borilgan tadqiqotlar natijasida qator dolzarb masalalar yechilgan.

Singulyar elliptik tipdagi tenglamaning fundamental yechimlari Gauss gipergeometrik funksiyasi yordamida ifodalangan va aralash tipdagi tenglama uchun chegaraviy masalalarni yechish nazariyasi ishlab chiqilgan.

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